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Regularity theory for elliptic and parabolic equations in nondivergence form

(비발산 타원 및 포물형 편미분 방정식의 정칙 이론)

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Regularity theory for elliptic and parabolic equations in nondivergence form

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Abstract

We investigate optimal regularity theory for nondivergence elliptic and parabolic equations with discontinuous coefficients in bounded domains. Global Hessian estimates of the solutions to the Dirichlet problems for such equations are obtained under the small bounded mean oscillation (BMO) condition of the coefficients in the setting of various function spaces such as weighted Lebesgue spaces, variable exponent Lebesgue spaces, weighted Orlicz spaces and weighted variable exponent Lebesgue spaces.

Key words: Regularity, nondivergence elliptic equation, nondivergence parabolic equation, strong solution, BMO space, weighted Lebesgue space, Orlicz space, variable exponent Lebesgue space
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Chapter 1

Introduction

This thesis is devoted to the study of global regularity theory for solutions to the nondivergence elliptic and parabolic equations with discontinuous coefficients in the setting of various function spaces such as weighted Lebesgue spaces, variable exponent spaces, weighted Orlicz spaces and weighted variable exponent spaces.

Firstly, we consider the following Dirichlet problem for second order elliptic equations in nondivergence form:

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and the matrix $\mathbf{A} = (a_{ij})$ of coefficients is assumed to be symmetric and uniformly elliptic; see (2.0.2). It is well known that there does not exist a unique strong solution in $W^{2,p}(\Omega)$ to (1.0.1) under the basic structural conditions on the coefficients like (2.0.2), even if the domain has an appropriate smoothness condition, as we see from [58, 61, 64]. It also turned out that this classical Dirichlet problem could not be solvable in an arbitrary bounded domain in $\mathbb{R}^n$ due to the famous examples of Zaremba and Lebesgue in [55, 76]. These facts naturally lead us to impose both a suitable additional condition on the coefficients and a certain geometric restriction on the boundary of the domain, in order to obtain the unique solvability of the problem (1.0.1) in $W^{2,p}(\Omega)$ for every $p \in (1, \infty)$.

As the classical results for the problem (1.0.1), if the coefficients a_{ij} are continuous and the boundary $\partial\Omega$ of the domain Ω belongs to C^2, then the estimate

$$\left\|D^2u\right\|_{L^p(\Omega)} \leq c\|f\|_{L^p(\Omega)}$$

is valid for every $1 < p < \infty$; see [39, 58]. In the case of discontinuous coefficients, Miranda [56] proved the well-posedness of (1.0.1) in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ when the coefficients a_{ij} belong to $W^{1,n}$ and $\partial\Omega$ is sufficiently regular. Since then, there have been further research activities on the $W^{2,p}$ regularity problem for (1.0.1) with discontinuous coefficients, and especially, in the papers [24] and [25], Chiarenza, Frasca and Longo proved the interior and boundary $W^{2,p}$ estimates of solutions to (1.0.1) when the coefficients a_{ij} have vanishing mean oscillation (VMO) and $\partial\Omega$ belongs to $C^{1,1}$. The approach in [24, 25] was mainly based on the explicit representation formulas involving singular integral operators and commutators. This approach was later generalized and applied by Palagachev, Di Fazio, Maugeri and Softova to the quasilinear elliptic problems, see [31, 57, 58, 62]. In [53], Krylov proposed a different approach for the $W^{2,p}$ solvability of solutions to the nondivergence type equations with VMO coefficients, which was mainly based upon the use of pointwise estimates of the sharp function of second order derivatives of solutions. Many studies on L^p regularity have been done via this approach as, for instance, in [32, 33, 46, 72]. There is another approach, the so-called maximal function free technique or large-M-inequality principle, which was introduced by Acerbi and Mingione [3] in order to prove the Calderón-Zygmund type estimates for parabolic systems of p-Laplacian type. This approach, not using either representation formulas or maximal functions, is suitable to the cases that a scaling in time and space is given differently such as p-Laplacian parabolic equations and systems, see, for instance, [8, 21, 34, 54].

Weighted $W^{2,p}$-estimates for second order elliptic equations We are concerned with weighted L^p regularity estimates for (1.0.1). More precisely, our goal is to find minimal conditions both on the coefficients a_{ij} and on the boundary $\partial\Omega$ of the domain under which we derive the global weighted $W^{2,p}$ estimate like

$$\left\|D^2 u\right\|_{L^p_w(\Omega)} \leq c\|f\|_{L^p_w(\Omega)}, \quad \forall p \in (2, \infty) \tag{1.0.2}$$

with a weight w belonging to the Muckenhoupt class $A_{\frac{p}{2}}$, where the constant $c > 0$ is independent of f and u.

In accordance with such research achievements on the L^p regularity, we focus on establishing the global weighted $W^{2,p}$ estimates for the Dirichlet problem (1.0.1), in particular, when coefficients a_{ij} have small BMO semi-norms and the domain Ω satisfies that its boundary $\partial\Omega$ belongs to $C^{1,1}$. Indeed, our results in this thesis can be considered as a natural extension of those in [25]. To be more exact, the L^p regularity of (1.0.1) in [25] is a special case of the weighted L^p regularity of (1.0.1) when a weight $w = 1$. Moreover, it is worth mentioning that the class of the coefficients which we are treating in this thesis, strictly contains VMO and so $W^{1,n}$, which were previously considered, for instance, in the works [9, 25, 53, 56].

Our approach in proving (1.0.2) is strongly influenced by [11, 12, 23, 71, 74]. Unlike the approaches in [3, 25, 53], we use the Hardy-Littlewood maximal function as the basic tool, to deduce the required power decay estimates for the weighted measure of the upper level sets for the maximal function of the second derivatives of the solutions. In particular, an essential part in our approach is to find a local estimate of solutions of the problem (1.0.1) by comparison with those of the limiting problems with constant coefficients of the local average values of the coefficients of (1.0.1). Furthermore, a weighted covering lemma and the standard flattening argument contribute largely to derive the required global weighted $W^{2,p}$ estimate along with interior and boundary weighted $W^{2,p}$ estimates.

$W^{2,p(\cdot)}$-estimates for second order elliptic equations We prove global $W^{2,p(\cdot)}$-estimates of the Dirichlet problem (1.0.1) for every variable exponent function $p : \mathbb{R}^n \to [0, \infty)$ with $1 < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty$ for some constants γ_1, γ_2. In particular, we are interested in the Calderón-Zygmund type estimate like

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \leq c\|f\|_{L^{p(\cdot)}(\Omega)} \tag{1.0.3}$$

where the constant c is independent of f and u. We need to impose appropriate regularity conditions on $p(\cdot)$, $\partial\Omega$ and $\mathbf{A}$ for the $W^{2,p(\cdot)}$-estimate (1.0.3) to be valid.

There have been rich research activities on regularity estimates for divergence type elliptic and parabolic problems in the frame of variable exponent function spaces, see [1, 2, 6, 7, 18, 19, 36] and references therein. On the other hand, little is known on regularity theory involved in variable exponent spaces to the nondivergence type problems even for the linear case. The nondivergence type problem (1.0.1) we are dealing with is used as a basic model in various areas such as probability and stochastic processes. The problem (1.0.1) can be regarded as linearizations to fully nonlinear PDEs, for instance, Hamilton-Jacobi-Bellman equation, which is associated with optimal control theory and stochastic differential game theory, see [66, 67]. It is also the Martingale problem which appears in application fields, like physics, engineering, economics and finance, see [45, 69]. We especially look at the case that the forcing term f belongs to the variable exponent Lebesgue space $L^{p(\cdot)}$. For some complex and sensitive materials or phenomena, the $L^{p(\cdot)}$-norm of f should have more reasonable information than its L^p-norm. This motivates us to find well-posedness in $W^{2,p(\cdot)}$ of (1.0.1) by essentially proving that

$$f \in L^{p(\cdot)}(\Omega) \Longrightarrow u \in W^{2,p(\cdot)}(\Omega)$$

along with the $W^{2,p(\cdot)}$-estimate (1.0.3), under possibly optimal conditions on $p(\cdot)$, $\mathbf{A}$ and $\partial\Omega$ to be necessarily imposed.

Our work is a natural extension of the $W^{2,p}$-estimate for any constant $p \in (1,\infty)$ to the $W^{2,p(\cdot)}$-estimate for any variable exponent $p(\cdot)$ mentioned earlier. The $W^{2,p(\cdot)}$-estimate has been known only for the Poisson equation. Indeed, Diening, Lengeler and Růžička in [30] showed the estimate (1.0.3) for the problem (1.0.1) when $\mathbf{A}$ is the identity matrix and $\partial\Omega \in C^{1,1}$, under the assumption that the given variable exponent $p(\cdot)$ has *log-Hölder continuity*, see (2.3.6). Here, the log-Hölder continuity of $p(\cdot)$ is essential in [30], since the proof is based on the boundedness of singular operators associated with the Poisson problem in $L^{p(\cdot)}$ space which is well understood if $p(\cdot)$ is log-Hölder continuous. In addition, crucial analysis properties for classical Lebesgue and Sobolev spaces, such as the boundedness of Hardy-Littlewood maximal operator, Sobolev type embeddings and the density of smooth functions, can be naturally extended to the variable exponent spaces under the condition (2.3.6) on a variable exponent function.

The proof of (1.0.3) is motivated from the so-called maximal function free technique, which has been explained earlier. We first derive the a priori interior and boundary $W^{2,p(\cdot)}$-estimates in small regions from comparison arguments on the upper-level sets of the second derivatives of solutions, and then by standard covering and flattening arguments, we establish the required global $W^{2,p(\cdot)}$-estimate. Our approach in proving (1.0.3) is suitable for resolving difficulties which should be caused by dealing with the wider function spaces, variable exponent spaces, than the classical Lebesgue and Sobolev spaces .

Secondly, we consider the following Dirichlet problem for second order parabolic equations in nondivergence form:

$$\begin{cases} u_t - a_{ij}D_{ij}u & = & f & \text{in} & \Omega_T, \\ u & = & 0 & \text{on} & \partial_p\Omega_T, \end{cases} \tag{1.0.4}$$

where Ω_T stands for the space-time cylinder $\Omega \times (0,T]$ over a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, and its parabolic boundary is denoted by $\partial_p\Omega_T := (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\})$. The coefficient matrix $\mathbf{A} = (a_{ij})$:

$\mathbb{R}^{n+1} \to \mathbb{R}^{n\times n}$ is assumed to be measurable, symmetric and uniformly parabolic; see (3.0.2).

Weighted estimates for second order parabolic equations in Orlicz spaces We derive global weighted Orlicz regularity estimates for solutions to (1.0.4). More precisely, our primary goal is to find minimal regularity assumptions on coefficients a_{ij} and boundary $\partial\Omega$ of domain Ω in order to establish the following global weighted Orlicz estimates:

$$\|u\|_{W^{2,1}L_w^\Phi(\Omega_T)} \leq c\|f\|_{L_w^\Phi(\Omega_T)}, \tag{1.0.5}$$

for some positive constant c being independent of f and u, where the given Young function Φ satisfies suitable conditions and the weight w belongs to some Muckenhoupt class.

Since the pioneering work of Chiarenza, Frasca and Longo in [24, 25], many studies have considered the regularity theory for elliptic and parabolic nondivergence form equations with discontinuous coefficients of bounded and vanishing mean oscillation types; see, for instance, [9, 10, 15, 58, 72, 75]. In particular, Bramanti and Cerutti [9] proved the $W_p^{2,1}$ solvability for (1.0.4) with VMO leading coefficients when $\partial\Omega \in C^{1,1}$. Our work can be regarded as a natural extension of the regularity results presented in [9] to weighted Orlicz spaces, which are more general than classical Sobolev spaces $W_p^{2,1}$. Specifically, if $\Phi(\rho) = \rho^{\frac{p}{2}}$ with $2 < p < \infty$ and $w(x,t) \equiv 1$, (1.0.5) is reduced to the $W_p^{2,1}$ estimates with $p > 2$ in [9].

Our approach in proving (1.0.5) which is based on the maximal function method is similar to that used in order to prove the weighted estimates (1.0.2) for the problem (1.0.1). We derive an appropriate power decay estimate of the weighted measure of the maximal function's upper level set for the first-order time derivatives and the second-order spatial derivatives of the solution, by means of Hardy-Littlewood maximal function's properties and a modified Vitali covering lemma. Especially, results on the index characterization of the weights in Orlicz spaces proved by Kerman and Torchinsky in [44] are helpful in resolving the primary difficulty that arises from features of weighted Orlicz spaces, which are much broader function spaces than those used in [9, 10, 58, 72, 75].

Weighted estimates for second order parabolic equations in variable exponent spaces For the Dirichlet problem (1.0.4), we prove the Calderón-Zygmund type estimates in the weighted variable exponent

Lebesgue spaces like

$$\begin{aligned}\|u_t\|_{L^{p(\cdot)}(\Omega_T,w)} &+ \|u\|_{L^{p(\cdot)}(\Omega_T,w)} \\ &+\|Du\|_{L^{p(\cdot)}(\Omega_T,w)} + \|D^2u\|_{L^{p(\cdot)}(\Omega_T,w)} \le c\|f\|_{L^{p(\cdot)}(\Omega_T,w)} \end{aligned} \tag{1.0.6}$$

for any log-Hölder continuous function $p(\cdot) : \mathbb{R}^{n+1} \to (1, \infty)$ with

$$1 < \inf_{z\in\mathbb{R}^{n+1}} p(z) \le \sup_{z\in\mathbb{R}^{n+1}} p(z) < \infty$$

for any weight w belonging to $A_{p(\cdot)}$ class and for some constant $c > 0$ independent of u and f, under a minimal regularity requirement on the coefficient matrix $\mathbf{A}$. The estimate (1.0.6) ultimately implies the weighted $L^{p(\cdot)}$ solvability of the equation (1.0.4) satisfying the implication

$$f \in L^{p(\cdot)}(\Omega_T, w) \implies u_t, D^2u \in L^{p(\cdot)}(\Omega_T, w).$$

The weighted variable exponent Lebesgue spaces have been actively studied, see [48, 49, 50, 65] and references therein. In particular, Diening and Hästö [27, 29] characterized the class of weights for which the maximal operator is bounded on the weighted variable exponent Lebesgue spaces, that is, the $A_{p(\cdot)}$ class which is a generalization of the classical Muckenhoupt class. On the other hand, there are not any results either of weighted $L^{p(\cdot)}$ estimates even for elliptic equations or of $L^{p(\cdot)}$ estimates for parabolic equations, even for the heat equation $u_t - \Delta u = f$.

Our method to prove (1.0.6) is influenced by the maximal function free technique that we have explained before. We first derive local interior and boundary *a priori* estimates. To do this, we apply a certain stopping time argument to find a suitable Vitali type covering of the upper-levels

$$\left\{z \in \Omega_T \cap Q : |u_t|^{\gamma_0 \frac{p(z)}{p^-}} + |D^2u|^{\gamma_0 \frac{p(z)}{p^-}} > \lambda\right\}$$

for sufficiently large numbers λ, where $\gamma_0 > 1$ is to be selected as a suitable constant satisfying $\gamma_0 \le \inf p(z)$ and $p^- := \inf_{z\in Q} p(z)$. We then estimate the weighted measures of these upper-level sets by taking advantage of the comparison estimates in the classical Lebesgue space L^{γ_0}, the log-Hölder continuity of the variable exponent $p(\cdot)$ and the properties of $A_{p(\cdot)}$ class. The desired estimate (1.0.6) follows by standard flattening and covering arguments along with an appropriate approximation procedure. We point out that in this procedure, we need to control the term $\|Du\|_{L^{p(\cdot)}(\Omega_T,w)}$.

When $p(\cdot) \equiv p$ and $w \equiv 1$ as the special case, this term can be easily controlled by $\|u\|_{L^p(\Omega_T)}$ and $\|D^2u\|_{L^p(\Omega_T)}$ from the interpolation inequality for the classical Sobolev space $W^{2,p}(\Omega)$. For the case of the weighted variable exponent Lebesgue space, however, it is not easy to do in a similar way as in the constant variable exponent case, because the exponent $p(\cdot)$ and the weight w depend on t variable. To overcome this difficulty, we instead use a certain compactness argument.

The remainder of this thesis is divided into two chapters. The first chapter (Chapter 2) contains weighted $W^{2,p}$- estimates and $W^{2,p(\cdot)}$-estimates for nondivergence elliptic equations. The second chapter (Chapter 3) deals with weighted Orlicz estimates and weighted $W^{2,1}_{p(\cdot)}$-estimates for nondivergence parabolic equations. The results in this thesis have been presented in four papers, already published or submitted. Two of them are joint works with Sun-Sig Byun [13, 14] and the other two are joint works with Sun-Sig Byun and Jihoon Ok [15, 16].

Chapter 2

Regularity theory for nondivergence elliptic equations

In this chapter, we consider the following Dirichlet problem:

$$\left\{ \begin{array}{rcccc} a_{ij}D_{ij}u & = & f & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{array} \right. \tag{2.0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and the matrix $\mathbf{A} = (a_{ij})$ of coefficients is assumed to be symmetric and uniformly elliptic; see (2.0.2). The purpose of this chapter is to derive Hessian estimates of the solutions to the problem (2.0.1) in setting of weighted Lebesgue spaces and variable exponent Lebesgue spaces.

We start this chapter with standard notations and definitions. For a point $y = (y', y_n) = (y_1, \cdots, y_{n-1}, y_n) \in \mathbb{R}^n$ and a number $r > 0$, let $B_r(y) = \{x \in \mathbb{R}^n : |x-y| < r\}$, $B_r^+(y) = B_r(y) \cap \{x_n > 0\}$ and $B_r'(y') = \{x' \in \mathbb{R}^{n-1} : |x' - y'| < r\}$. For the sake of simplicity, we write $B_r = B_r(0)$ and $B_r^+ = B_r^+(0)$. We also denote $T_r(y) = B_r(y) \cap \{x_n = 0\}$ and $T_r = B_r \cap \{x_n = 0\}$. For a vector valued function $\mathbf{f} : U \to \mathbb{R}^N$, where U is a bounded domain in $\mathbb{R}^n$ and $N \geq 1$, we denote $\bar{\mathbf{f}}_U$ by the integral average of $\mathbf{f}$ on U, that is,

$$\bar{\mathbf{f}}_U = \fint_U \mathbf{f}(x)dx = \frac{1}{|U|}\int_U \mathbf{f}(x)dx.$$

In this chapter, we always assume that the coefficient matrix $\mathbf{A} = (a_{ij})$:

$\mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a bounded, measurable and matrix-valued function on $\mathbb{R}^n$ with the symmetric condition $a_{ij} = a_{ji}$. In addition, $\mathbf{A} = (a_{ij})$ is supposed to be *uniformly elliptic* with the ellipticity constant $\Lambda \geq 1$, that is,

$$\Lambda^{-1}|\xi|^2 \ \leq\ \langle \mathbf{A}(x)\xi, \xi\rangle \leq\ \Lambda|\xi|^2 \quad \text{for}\ \ \forall \xi \in \mathbb{R}^n \text{ and a.e. } x \in \mathbb{R}^n. \tag{2.0.2}$$

Next, we introduce our principal assumption on the coefficient matrix $\mathbf{A} = (a_{ij})$.

Definition 2.0.1. For $\delta, R > 0$, we say that the coefficient matrix $\mathbf{A} = (a_{ij})$ is (δ, R)-*vanishing* if

$$\sup_{0<r\leq R}\ \sup_{y\in\mathbb{R}^n} \left(\fint_{B_r(y)} |\mathbf{A}(x) - \overline{\mathbf{A}}_{B_r(y)}|^2 \, dx \right)^{\frac{1}{2}} \leq \delta, \tag{2.0.3}$$

In the above definition, R can be any positive number by scaling the given equations, whereas δ is invariant under such scaling. A locally integrable function f is called *of bounded mean oscillation on* $\mathbb{R}^n$, denoted by $f \in \mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_* := \sup_{B\subset\mathbb{R}^n} \fint_B \left|f - \overline{f}_B\right| \, dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. In this chapter, we assume that $\mathbf{A} = (a_{ij})$ is in the John-Nirenberg space BMO of functions of bounded mean oscillation with small BMO seminorms, which we defined above in (2.0.3). This is a more general concept than the VMO condition appeared in other papers such as [25] and [53]. Since the coefficients a_{ij} can be extended in $\mathbb{R}^n$ preserving the small BMO condition (see [4]), we can consider the small BMO coefficients a_{ij} to be defined in $\mathbb{R}^n$ throughout this thesis. Moreover, we notice that the condition (2.0.3) is equivalent to the small BMO condition $\|\mathbf{A}\|_* \leq \delta$ by the John-Nirenberg inequality (see [42] for details). Therefore, we also use

$$[\mathbf{A}]_R := \sup_{0<r\leq R}\ \sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} |\mathbf{A}(x) - \overline{\mathbf{A}}_{B_r(y)}| \, dx \leq \delta, \tag{2.0.4}$$

as the definition of (δ, R)-*vanishing* of the coefficient matrix $\mathbf{A} = (a_{ij})$ instead of (2.0.3).

2.1 Preliminary results

In this section, we present comparison estimates in L^q spaces, $1 < q < \infty$ that will play crucial roles in proving the main results of this chapter. To prove the comparison estimates, we shall adapt a compactness argument.

We first derive the comparison estimates in L^2 spaces, that will be used in the proofs of the interior and boundary weighted estimates for (2.0.1) in Chapters 2.2.2 and 2.2.3.

Lemma 2.1.1. *For any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon, n, \Lambda) > 0$ so that if $u \in W^{2,2}(B_6)$ is a solution of (2.2.4) with*

$$\fint_{B_4} |D^2 u|^2 dx \le 1 \quad \text{and} \fint_{B_4} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4}|^2 dx \le \delta^2,$$

then there exist a constant matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})$ with $\|\overline{\mathbf{A}}_{B_4} - \tilde{\mathbf{A}}\|_{L^\infty(\mathbb{R}^n)} \le \epsilon$ and a solution $v \in W^{2,2}(B_4)$ of

$$\tilde{a}_{ij} D_{ij} v = 0 \quad \text{in } B_4, \tag{2.1.1}$$

with

$$\fint_{B_4} |D^2 v|^2 dx \le 1 \tag{2.1.2}$$

such that

$$\int_{B_4} \left| u - \overline{u}_{B_4} - (\overline{Du})_{B_4} \cdot x - v \right|^2 dx \le \epsilon^2.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $\{u_k\}_{k=1}^\infty$, $\{f_k\}_{k=1}^\infty$ and $\{\mathbf{A}_k\}_{k=1}^\infty = \{(a_{ij}^k)\}_{k=1}^\infty$ such that $u_k \in W^{2,2}(B_6)$ is a solution of

$$a_{ij}^k D_{ij} u_k = f_k \quad \text{in } B_6,$$

with

$$\fint_{B_4} |D^2 u_k|^2 dx \le 1 \quad \text{and} \fint_{B_4} |f_k|^2 + |\mathbf{A}_k - \overline{\mathbf{A}_k}_{B_4}|^2 dx \le \frac{1}{k^2}, \tag{2.1.3}$$

but

$$\int_{B_4} \left| u_k - \overline{u_k}_{B_4} - (\overline{Du_k})_{B_4} \cdot x - v \right|^2 dx > \epsilon_0^2, \tag{2.1.4}$$

for any constant matrix $\tilde{\mathbf{A}}$ with $\|\overline{\mathbf{A}}_{B_4} - \tilde{\mathbf{A}}\|_{L^\infty(\mathbb{R}^n)} \le \epsilon_0$ and any solution $v \in W^{2,2}(B_4)$ of (2.1.9) satisfying (2.1.2).

Set $w_k := u_k - \overline{u_k}_{B_4} - (\overline{Du_k})_{B_4} \cdot x$. From the fact $\overline{w_k}_{B_4} = 0$, we then use the Poincaré inequality twice to discover

$$\begin{aligned} ⨍_{B_4} |w_k|^2 dx &\leq c ⨍_{B_4} |Dw_k|^2 dx = c ⨍_{B_4} \left|Du_k - (\overline{Du_k})_{B_4}\right|^2 dx \quad (2.1.5) \\ &\leq c ⨍_{B_4} |D^2 u_k|^2 dx \leq c, \end{aligned}$$

where the last inequality comes from (2.1.3). Therefore it follows that

$$\|w_k\|_{W^{2,2}(B_4)} \leq c$$

for some positive constant $c = c(n, \Lambda)$, and so there exist a subsequence of $\{w_k\}_{k=1}^{\infty}$, which we still denote by $\{w_k\}_{k=1}^{\infty}$, and a function $w_0 \in W^{2,2}(B_4)$ such that

$$w_k \rightharpoonup w_0 \text{ weakly in } W^{2,2}(B_4) \text{ and } w_k \to w_0 \text{ strongly in } L^2(B_4) \qquad (2.1.6)$$

as $k \to \infty$. Furthermore, from (2.1.3) and (2.1.6) that

$$⨍_{B_4} |D^2 w_0|^2 dx \leq \liminf_{k\to\infty} |D^2 w_k|^2 dx \leq 1.$$

Since $\{\overline{\mathbf{A}_k}_{B_4}\}$ is uniformly bounded in $\mathbb{R}^n$, it also has a subsequence, which is denoted by $\{\overline{\mathbf{A}_k}\}$, such that $\|\overline{\mathbf{A}_k} - \mathbf{A}_0\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $k \to \infty$ for some constant matrix $\mathbf{A}_0 = (a_{ij}^0)$. Then by (2.1.3), we have

$$\mathbf{A}_k \to \mathbf{A}_0 \quad \text{in } L^2(B_4) \quad \text{as } k \to \infty. \qquad (2.1.7)$$

From (2.1.3), (2.1.6) and (2.1.7), one can readily check that $w_0 \in W^{2,2}(B_4)$ is a solution of

$$a_{ij}^0 D_{ij} w_0 = 0 \quad \text{in} \quad B_4.$$

However, recalling (2.1.6), it is a contradiction to the inequality (2.1.4). This completes the proof. □

Corollary 2.1.2. *Under the hypotheses and conclusion of Lemma 2.1.1, we have*

$$⨍_{B_2} \left|D^2(u - v)\right|^2 dx \leq \epsilon^2.$$

Proof. Apply Lemma 2.1.1 to η and $\delta(\eta, n, \Lambda)$ replaced by ϵ and $\delta(\epsilon, n, \Lambda)$ respectively, to get that there are a constant matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})$ and a solution

$v \in W^{2,2}(B_4)$ of (2.1.1) such that

$$\fint_{B_4} |D^2 v|^2 dx \leq 1 \text{ and } \fint_{B_4} \left|u - \overline{u}_{B_4} - (\overline{Du})_{B_4} \cdot x - v\right|^2 dx \leq \eta^2,$$

provided

$$\fint_{B_4} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4}|^2 dx \leq \delta^2.$$

Then using $C^{1,1}$ regularity for (2.1.1), we obtain

$$\|D^2 v\|_{L^\infty(B_3)}^2 \leq c \fint_{B_4} |D^2 v|^2 dx \leq c,$$

for some positive constant $c = c(n, \Lambda)$.

One can easily see that $u - \overline{u}_{B_4} - (\overline{Du})_{B_4} \cdot x - v \in W^{2,2}(B_3)$ is a solution of

$$a_{ij} D_{ij} \left(u - \overline{u}_{B_4} - (\overline{Du})_{B_4} \cdot x - v\right) = f - (a_{ij} - \tilde{a}_{ij}) D_{ij} v \quad \text{in} \quad B_3,$$

Then from Lemma 2.2.9, we deduce for some constant $c = c(n, \Lambda) > 0$,

$$\begin{aligned}
&\fint_{B_2} |D^2(u - v)|^2 dx \\
&\leq c \left(\fint_{B_3} |f - (a_{ij} - \tilde{a}_{ij}) D_{ij} v|^2 dx + \fint_{B_3} |u - \overline{u}_{B_4} - (\overline{Du})_{B_4} \cdot x - v|^2 dx \right) \\
&\leq c \left(\fint_{B_4} |f|^2 dx + \|D^2 v\|_{L^\infty(B_3)}^2 \fint_{B_4} |a_{ij} - \tilde{a}_{ij}|^2 \, dx + \frac{\eta^2}{|B_3|} \right) \\
&\leq c(\delta^2 + \eta^2) \leq \epsilon^2,
\end{aligned}$$

if we take η and δ satisfying the last inequality. This finishes the proof. □

Lemma 2.1.3. *For any $\epsilon > 0$, there is a small $\delta = \delta(\epsilon, n, \Lambda) > 0$ so that if $u \in W^{2,2}(B_6^+)$ is a solution of*

$$\begin{cases} a_{ij} D_{ij} u &= f \quad \text{in} \quad B_6^+, \\ u &= 0 \quad \text{on} \quad T_6, \end{cases} \tag{2.1.8}$$

with

$$\fint_{B_4^+} |D^2 u|^2 dx \leq 1 \quad \text{and} \quad \fint_{B_4^+} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4^+}|^2 dx \leq \delta^2,$$

then there exist a constant matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})$ with $\|\overline{\mathbf{A}}_{B_4^+} - \tilde{\mathbf{A}}\|_{L^\infty(\mathbb{R}^n)} \leq \epsilon$

and a solution $v \in W^{2,2}(B_4^+)$ *of*

$$\begin{cases} \tilde{a}_{ij} D_{ij} v = 0 & \text{in} \quad B_4^+, \\ v = 0 & \text{on} \quad T_4, \end{cases} \tag{2.1.9}$$

with

$$⨍_{B_4^+} |D^2 v|^2 dx \le 1$$

such that

$$\int_{B_4^+} \left| u - (\overline{D_n u})_{B_4^+} x_n - v \right|^2 dx \le \epsilon^2.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $\{u_k\}_{k=1}^\infty$, $\{f_k\}_{k=1}^\infty$ and $\{\mathbf{A}_k\}_{k=1}^\infty = \{(a_{ij}^k)\}_{k=1}^\infty$ such that $u_k \in W^{2,2}(B_6^+)$ is a solution of

$$\begin{cases} a_{ij}^k D_{ij} u_k = f_k & \text{in} \quad B_6^+, \\ u_k = 0 & \text{on} \quad T_6, \end{cases}$$

with

$$⨍_{B_4^+} |D^2 u_k|^2 dx \le 1 \quad \text{and} ⨍_{B_4^+} |f_k|^2 + |\mathbf{A}_k - \overline{\mathbf{A}_k}_{B_4^+}|^2 dx \le \frac{1}{k^2}, \tag{2.1.10}$$

but

$$\int_{B_4^+} \left| u_k - (\overline{D_n u_k})_{B_4^+} x_n - v \right|^2 dx > \epsilon_0^2, \tag{2.1.11}$$

for any constant matrix $\tilde{\mathbf{A}}$ with $\|\overline{\mathbf{A}}_{B_4^+} - \tilde{\mathbf{A}}\|_{L^\infty(\mathbb{R}^n)} \le \epsilon_0$ and any solution $v \in W^{2,2}(B_4^+)$ of (2.1.9) satisfying

$$⨍_{B_4^+} |D^2 v|^2 dx \le 1.$$

We write $w_k := u_k - (\overline{D_n u_k})_{B_4^+} x_n$ and claim

$$\|w_k\|_{W^{2,2}(B_4^+)} \le c \tag{2.1.12}$$

for some positive constant $c = c(n, \Lambda)$. To do this, recalling $D_i u_k = 0$ on T_4 for $1 \le i \le n-1$, we use Poincaré inequality and (2.1.10) to find that for some $c = c(n) > 0$,

$$⨍_{B_4^+} |D_i(w_k)|^2 \, dx \le c ⨍_{B_4^+} |D_i(u_k)|^2 \, dx \le c ⨍_{B_4^+} \left|D^2 u_k\right|^2 dx \le c$$

for $1 \le i \le n-1$. Moreover, we see that

$$\fint_{B_4^+} |D_n(w_k)|^2\,dx = \fint_{B_4^+} \left| D_n u_k - (\overline{D_n u_k})_{B_4^+} \right|^2 dx \le c \fint_{B_4^+} |D^2 u_k|^2 dx \le c$$

for some constant $c = c(n) > 0$. Thus, we have that for some positive constant $c = c(n, \Lambda)$,

$$\fint_{B_4^+} |Dw_k|^2\,dx \le c. \tag{2.1.13}$$

But then, since $w_k = 0$ in T_4, it follows from the Poincaré inequality and (2.1.13) that for some $c = c(n) > 0$,

$$\fint_{B_4^+} |w_k|^2 dx \le c \fint_{B_4^+} |Dw_k|^2\,dx \le c. \tag{2.1.14}$$

We next recall (2.1.10) to see that

$$\fint_{B_4^+} \left|D^2 w_k\right|^2 dx = \fint_{B_4^+} \left|D^2 u_k\right|^2 dx \le 1. \tag{2.1.15}$$

Then the claim (2.1.12) follows from (2.1.13), (2.1.14) and (2.1.15). Consequently, there exist a subsequence of $\{w_k\}_{k=1}^{\infty}$, which we still denote by $\{w_k\}_{k=1}^{\infty}$, and a function $w_0 \in W^{2,2}(B_4^+)$ such that

$$w_k \rightharpoonup w_0 \text{ weakly in } W^{2,2}(B_4^+) \text{ and } w_k \to w_0 \text{ strongly in } L^2(B_4^+). \tag{2.1.16}$$

In addition, it follows from (2.1.15) and (2.1.16) that

$$\fint_{B_4^+} |D^2 w_0|^2 dx \le 1. \tag{2.1.17}$$

Since $\left\{\overline{\mathbf{A}_k}_{B_4^+}\right\}$ is uniformly bounded in $L^\infty(B_4^+)$, it also has a subsequence, which is denoted by $\left\{\overline{\mathbf{A}_k}\right\}$, such that $\|\overline{\mathbf{A}_k} - \mathbf{A}_0\|_{L^\infty(B_4^+)} \to 0$ as $k \to \infty$ for some constant matrix $\mathbf{A}_0 = (a_{ij}^0)$. Then by (2.1.10), we have

$$\mathbf{A}_k \to \mathbf{A}_0 \ \text{ in } L^2(B_4^+) \ \text{ as } k \to \infty. \tag{2.1.18}$$

From (2.1.10), (2.1.16) and (2.1.18), it is easy to check that $w_0 \in W^{2,2}(B_4^+)$ is a solution of

$$\begin{cases} a_{ij}^0 D_{ij} w_0 &= 0 \quad \text{in} \quad B_4^+, \\ w_0 &= 0 \quad \text{on} \quad T_4. \end{cases}$$

We then recall (2.1.16) and (2.1.17) to reach a contradiction to the inequality (2.1.11). This completes the proof. $\square$

Corollary 2.1.4. *Under the hypotheses and conclusion of Lemma 2.1.3, we have*

$$\fint_{B_2^+} \left|D^2(u-v)\right|^2 dx \le \epsilon^2.$$

Proof. We apply Lemma 2.1.3 to η and $\delta(\eta, n, \Lambda)$ replaced by ϵ and $\delta(\epsilon, n, \Lambda)$ respectively, to find that there is a solution $v \in W^{2,2}(B_4^+)$ of (2.1.9) such that

$$\fint_{B_4^+} \left|D^2 v\right|^2 dx \le 1 \text{ and } \int_{B_4^+} \left|u - (\overline{D_n u})_{B_4^+} x_n - v\right|^2 dx \le \eta^2,$$

provided that

$$\fint_{B_4^+} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4^+}|^2 dx \le \delta^2.$$

Then by $C^{1,1}$ regularity for (2.1.9) up to the flat boundary, we discover that

$$\|D^2 v\|\|^2_{L^\infty(B_3^+)} \le c \fint_{B_4^+} |D^2 v|^2 dx \le c,$$

for some positive constant $c = c(n, \Lambda)$.

We next observe that $u - (\overline{D_n u})_{B_4^+} x_n - v \in W^{2,2}(B_3^+)$ is a solution of

$$\begin{cases} a_{ij} D_{ij}\left(u - (\overline{D_n u})_{B_4^+} x_n - v\right) = f - (a_{ij} - \tilde{a}_{ij}) D_{ij} v & \text{in} \quad B_3^+, \\ u - (\overline{D_n u})_{B_4^+} x_n - v = 0 & \text{on} \quad T_3. \end{cases}$$

Then, according to Lemma 2.2.15, we compute for some constant $c = c(n, \Lambda) > 0$,

$$\begin{aligned} &\fint_{B_2^+} |D^2(u-v)|^2 dx \\ &\le c\left(\fint_{B_3^+} |f - (a_{ij} - \tilde{a}_{ij})\, D_{ij} v|^2\, dx + \fint_{B_3^+} \left|u - (\overline{D_n u})_{B_4^+} x_n - v\right|^2 dx\right) \\ &\le c\left(\fint_{B_4^+} |f|^2 dx + \|D^2 v\|^2_{L^\infty(B_3^+)} \fint_{B_4^+} |a_{ij} - \tilde{a}_{ij}|^2\, dx + \frac{\eta^2}{|B_3^+|}\right) \\ &\le c(\delta^2 + \eta^2) \le \epsilon^2, \end{aligned}$$

if we take η and δ satisfying the last inequality. This completes the proof. $\square$

The following are the comparison estimates in L^q spaces for $1 < q < \infty$, that will be essential in proving the $W^{2,p(\cdot)}$-estimates for (2.0.1) in Chapter 2.3. When $q = 2$, these are equivalent to the comparison estimates in Lemmas 2.1.1 and 2.1.3. In the proofs of Lemma 2.1.5, Corollary 2.1.6 and Lemma 2.1.7, the constant c is any positive constant depending only on n, Λ and q.

Lemma 2.1.5. *Let $1 < q < \infty$ and let $\mathbf{B} = (b_{ij}) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfy the uniformly elliptic condition* (2.0.2) *with $\mathbf{A}$ replaced by $\mathbf{B}$. For any $\epsilon \in (0,1)$, there is $\delta = \delta(\epsilon, n, \Lambda, q) > 0$ so that if $\mathbf{B}$ is $(\delta, 4)$-vanishing and if $w \in W^{2,q}(B_4)$ is a solution of*

$$b_{ij} D_{ij} w = g \quad \text{in } B_4 \tag{2.1.19}$$

with

$$\fint_{B_4} |D^2 w|^q dx \leq 1 \quad \text{and} \quad \fint_{B_4} |g|^q dx \leq \delta,$$

then there exists a solution $v \in W^{2,q}(B_3)$ of

$$\overline{b_{ij}}_{B_4} D_{ij} v = 0 \quad \text{in } B_3 \tag{2.1.20}$$

with

$$\fint_{B_3} |D^2 v|^q dx \leq 2^{q-1+n}, \tag{2.1.21}$$

such that

$$\int_{B_3} |w - \overline{w}_{B_4} - (\overline{Dw})_{B_4} \cdot x - v|^q dx \leq \epsilon.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $w_l \in W^{2,q}(B_4)$, $g_l \in L^q(B_4)$ and $\mathbf{B}_l = (b^l_{ij}) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, where $l = 0, 1, 2, \ldots$, such that $\mathbf{B}_l$ is uniformly elliptic with the ellipticity constant Λ and $[\mathbf{B}_l]_4 \leq \frac{1}{l}$, which implies that

$$\fint_{B_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{l B_4}| dx \leq \frac{1}{l}, \tag{2.1.22}$$

and $w_l \in W^{2,q}(B_4)$ is a solution of

$$b^l_{ij} D_{ij} w_l = g_l \quad \text{in } B_4, \tag{2.1.23}$$

with

$$\fint_{B_4} |D^2 w_l|^q dx \le 1 \quad \text{and} \quad \fint_{B_4} |g_l|^q dx \le \frac{1}{l}, \tag{2.1.24}$$

but

$$\int_{B_3} |w_l - \overline{w_l}_{B_4} - (\overline{Dw_l})_{B_4} \cdot x - v|^q dx > \epsilon_0, \tag{2.1.25}$$

for any solution $v \in W^{2,q}(B_3)$ of

$$\overline{b_{ij}^l}_{B_4} D_{ij} v = 0 \quad \text{in } B_3, \tag{2.1.26}$$

satisfying (2.1.21).

Setting $h_l = w_l - \overline{w_l}_{B_4} - (\overline{Dw_l})_{B_4} \cdot x$, we obtain

$$\|h_l\|_{W^{2,q}(B_4)} \le c. \tag{2.1.27}$$

Indeed, using the Poincaré inequality, it follows from (2.1.24) that

$$\fint_{B_4} |Dh_l|^q dx = \fint_{B_4} |Dw_l - (\overline{Dw_l})_{B_4}|^q dx \le c \fint_{B_4} |D^2 w_l|^q dx \le c$$

and moreover, since $\overline{h_l}_{B_4} = 0$,

$$\fint_{B_4} |h_l|^q dx \le c \fint_{B_4} |Dh_l|^q dx \le c.$$

Therefore (2.1.27) holds, and so there exist a subsequence of $\{h_l\}_{l=1}^{\infty}$, which is still denoted by $\{h_l\}_{l=1}^{\infty}$, and a function $h_0 \in W^{2,q}(B_4)$ such that

$$\begin{cases} h_l \rightharpoonup h_0 & \text{weakly in } W^{2,q}(B_4), \\ h_l \to h_0 & \text{strongly in } L^q(B_4), \end{cases} \quad \text{as } l \to \infty. \tag{2.1.28}$$

From the uniformly ellipticity of $\mathbf{B}_l$ and (2.1.22), we derive

$$\fint_{B_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{lB_4}|^{q'} dx \le (2\Lambda)^{q'-1} \fint_{B_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{lB_4}| dx \le \frac{(2\Lambda)^{q'-1}}{l},$$

where $q' = \frac{q}{q-1}$. Recalling that $\{\overline{\mathbf{B}}_{lB_4}\}_{l=1}^{\infty}$ is bounded in $\mathbb{R}^{n\times n}$, it also has a subsequence, which is still denoted by $\{\overline{\mathbf{B}}_{lB_4}\}$, such that

$$\overline{\mathbf{B}}_{lB_4} \longrightarrow \mathbf{B}_0 \ \text{ in } \ \mathbb{R}^{n\times n} \quad \text{as } l \to \infty, \tag{2.1.29}$$

for some constant matrix $\mathbf{B}_0 = (b_{ij}^0)$. Hence, we have

$$\mathbf{B}_l \longrightarrow \mathbf{B}_0 \ \text{ in } \ L^{q'}(B_4) \quad \text{as} \ \ l \to \infty. \tag{2.1.30}$$

By the definition of h_l, we have from (2.1.23) that

$$\int_{B_4} b_{ij}^l D_{ij} h_l \varphi dx = \int_{B_4} g_l \varphi dx$$

for every $\varphi \in C_0^\infty(B_4)$. Letting $l \to \infty$, in view of (2.1.24), (2.1.28) and (2.1.30), we deduce that

$$\int_{B_4} b_{ij}^0 D_{ij} h_0 \varphi dx = 0,$$

and hence, $h_0 \in W^{2,q}(B_4)$ is a solution of

$$b_{ij}^0 D_{ij} h_0 = 0 \quad \text{in} \ \ B_4.$$

Furthermore, from (2.1.24) and (2.1.28), we get

$$⨍_{B_4} |D^2 h_0|^q dx \le \liminf_{l\to\infty} ⨍_{B_4} |D^2 h_l|^q dx \le 1. \tag{2.1.31}$$

By the $W^{2,q}$ regularity theory for elliptic equations with constant coefficients (see [39]), there exists the unique solution $v_l \in W^{2,q}(B_3)$ of

$$\begin{cases} \overline{b_{ij}^l}_{B_4} D_{ij} v_l &= 0 \quad \text{in} \quad B_3, \\ v_l &= h_0 \quad \text{on} \quad \partial B_3, \end{cases}$$

and then $v_l - h_0$ is the unique solution of

$$\begin{cases} \overline{b_{ij}^l}_{B_4} D_{ij}(v_l - h_0) &= (b_{ij}^0 - \overline{b_{ij}^l}_{B_4}) D_{ij} h_0 \quad \text{in} \quad B_3, \\ v_l - h_0 &= 0 \quad \text{on} \quad \partial B_3 \end{cases}$$

with the estimate

$$\|v_l - h_0\|_{W^{2,q}(B_3)} \le c|\mathbf{B}_0 - \overline{\mathbf{B}_l}_{B_4}| \|D^2 h_0\|_{L^q(B_3)} \le c|\mathbf{B}_0 - \overline{\mathbf{B}_l}_{B_4}|,$$

where the last inequality comes from (2.1.31). In view of (2.1.29), v_l converges strongly to h_0 in $W^{2,q}(B_3)$.

Hence,

$$\begin{aligned}&\fint_{B_3} |D^2 v_l|^q dx \\ &\quad \le 2^{q-1} \left(\fint_{B_3} |D^2 (v_l - h_0)|^q dx + \fint_{B_3} |D^2 h_0|^q dx \right) \le 2^{q-1+n}\end{aligned}$$

for sufficiently large l, and

$$\begin{aligned}\|w_l - \overline{w_l}_{B_4} - (\overline{Dw_l})_{B_4} \cdot x - v_l\|_{L^q(B_3)} &= \|h_l - v_l\|_{L^q(B_3)} \\ &\le \|h_l - h_0\|_{L^q(B_3)} + \|h_0 - v_l\|_{L^q(B_3)} \longrightarrow 0,\end{aligned}$$

as $l \to \infty$, by (2.1.28). This is a contradiction to (2.1.25). □

Corollary 2.1.6. *Under the hypotheses and conclusion of Lemma 2.1.5, we have*

$$\fint_{B_1} |D^2(w - v)|^q dx \le \epsilon.$$

Proof. From the assumptions of Lemma 2.1.5, we have

$$\fint_{B_4} |g|^q dx \le \delta \quad \text{and} \quad \fint_{B_4} |\mathbf{B} - \overline{\mathbf{B}}_{B_4}| dx \le \delta. \tag{2.1.32}$$

We apply Lemma 2.1.5 to η in order to discover that there is a solution $v \in W^{2,q}(B_3)$ of (2.1.19) such that

$$\fint_{B_3} |D^2 v|^q dx \le 2^{q-1+n} \text{ and } \int_{B_3} |w - \overline{w}_{B_4} - (\overline{Dw})_{B_4} \cdot x - v|^q dx \le \eta, \tag{2.1.33}$$

by taking sufficiently small $\delta = \delta(\eta, n, \Lambda, q) > 0$. Then we use the local $C^{1,1}$ regularity for (2.1.20) to obtain

$$\|D^2 v\|^q_{L^\infty(B_2)} \le c \fint_{B_3} |D^2 v|^q dx \le c. \tag{2.1.34}$$

We notice that $w - \overline{w}_{B_4} - (\overline{Dw})_{B_4} \cdot x - v \in W^{2,q}(B_3)$ is a solution of

$$b_{ij} D_{ij} \left(w - \overline{w}_{B_4} - (\overline{Dw})_{B_4} \cdot x - v \right) = g - (b_{ij} - \overline{b_{ij}}_{B_4}) D_{ij} v \quad \text{in } B_3.$$

Taking $\delta = \delta(\eta, n, \Lambda, q) > 0$ sufficiently small to apply Lemma 2.3.3 to the above equation, we deduce from (2.0.2), (2.1.32), (2.1.33) and (2.1.34) that

$$
\begin{aligned}
&\fint_{B_1} \left|D^2(w-v)\right|^q dx \\
&\le c\left(\fint_{B_2} \left|g-(b_{ij}-\overline{b_{ij}}_{B_4})D_{ij}v\right|^q dx \right. \\
&\qquad \left. +\fint_{B_2} \left|w-\overline{w}_{B_4}-(\overline{Dw})_{B_4}\cdot x - v\right|^q dx\right) \\
&\le c\left(\fint_{B_4} |g|^q dx + \|D^2 v\|^q_{L^\infty(B_2)} \fint_{B_4} \left|\mathbf{B}-\overline{\mathbf{B}}_{B_4}\right|^q dx + \eta\right) \\
&\le c\left(\delta + (2\Lambda)^{q-1} \fint_{B_4} \left|\mathbf{B}-\overline{\mathbf{B}}_{B_4}\right| dx + \eta\right) \\
&\le c(\delta+\eta).
\end{aligned}
$$

Finally, choosing $\eta = \eta(\epsilon, n, \Lambda, q) > 0$ and $\delta = \delta(\epsilon, n, \Lambda, q) > 0$ sufficiently small, we complete the proof of the lemma. □

We next derive the comparison estimate on a half ball, which will be deduced in an analogous way to Lemma 2.1.5.

Lemma 2.1.7. *Let $1 < q < \infty$ and let $\mathbf{B} = (b_{ij}) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfy the uniformly elliptic condition (2.0.2) with $\mathbf{A}$ replaced by $\mathbf{B}$. For any $\epsilon \in (0,1)$, there is $\delta = \delta(\epsilon, n, \Lambda, q) > 0$ so that if $\mathbf{B}$ is $(\delta, 4)$-vanishing and if $w \in W^{2,q}(B_4^+)$ is a solution of*

$$
\begin{cases} b_{ij}D_{ij}w = g & \text{in} \quad B_4^+, \\ w = 0 & \text{on} \quad T_4 \end{cases} \tag{2.1.35}
$$

with

$$
\fint_{B_4^+} |D^2 w|^q dx \le 1 \quad \text{and} \quad \fint_{B_4^+} |g|^q dx \le \delta,
$$

then there exists a solution $v \in W^{2,q}(B_3^+)$ of

$$
\begin{cases} \overline{b_{ij}}_{B_4^+} D_{ij}v = 0 & \text{in} \quad B_3^+, \\ v = 0 & \text{on} \quad T_3 \end{cases}
$$

with

$$
\fint_{B_3^+} |D^2 v|^q dx \le 2^{q-1+n} \tag{2.1.36}
$$

such that

$$\int_{B_3^+} |w - (\overline{D_n w})_{B_4^+} \cdot x_n - v|^q dx \le \epsilon.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $w_l \in W^{2,q}(B_4^+)$, $g_l \in L^q(B_4^+)$ and $\mathbf{B}_l = (b_{ij}^l) : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, where $l = 0, 1, 2, \dots$, such that $\mathbf{B}_l$ is uniformly elliptic with the ellipticity constant Λ and $[\mathbf{B}_l]_4 \le \frac{1}{l}$, which implies that

$$\begin{aligned}
&\fint_{B_4^+} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{B_4^+}| dx \\
&\le 2 \fint_{B_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{B_4}| dx + |\overline{\mathbf{B}_l}_{B_4^+} - \overline{\mathbf{B}_l}_{B_4}| \\
&\le 2 \fint_{B_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{B_4}| dx + 2 \fint_{B_4} |\mathbf{B}_l - \overline{\mathbf{B}_l}_{B_4}| dx \le \frac{4}{l} \qquad (2.1.37)
\end{aligned}$$

and $w_l \in W^{2,q}(B_4^+)$ is a solution of

$$\begin{cases} b_{ij}^l D_{ij} w_l = g_l & \text{in} \quad B_4^+, \\ w_l = 0 & \text{on} \quad T_4 \end{cases}$$

with

$$\fint_{B_4^+} |D^2 w_l|^q dx \le 1 \quad \text{and} \quad \fint_{B_4^+} |g_l|^q \le \frac{1}{l}, \qquad (2.1.38)$$

but

$$\int_{B_3^+} |w_l - (\overline{D_n w_l})_{B_4^+} \cdot x_n - v|^q dx > \epsilon_0, \qquad (2.1.39)$$

for any solution $v \in W^{2,q}(B_3^+)$ of

$$\begin{cases} \overline{b_{ij}^l}_{B_4^+} D_{ij} v = 0 & \text{in} \quad B_3^+, \\ v = 0 & \text{on} \quad T_3, \end{cases}$$

satisfying (2.1.36).

Set $h_l = w_l - (\overline{D_n w_l})_{B_4^+} \cdot x_n$. Let us now claim

$$\|h_l\|_{W^{2,q}(B_4^+)} \le c. \qquad (2.1.40)$$

To do this, we first note that $D_i w_l = 0$ on T_4 for $1 \le i \le n-1$ to deduce from the Poincaré inequality and (2.1.38) that

$$\fint_{B_4^+} |D_i(h_l)|^q dx \leq c \fint_{B_4^+} |D_i(w_l)|^q dx \leq c \fint_{B_4^+} |D^2 w_l|^q dx \leq c,$$

for $1 \leq i \leq n-1$, and moreover,

$$\begin{aligned} \fint_{B_4^+} |D_n(h_l)|^q dx &= \fint_{B_4^+} |D_n w_l - (\overline{D_n w_l})_{B_4^+}|^q dx \\ &\leq c \fint_{B_4^+} |D^2 w_l|^q dx \leq c. \end{aligned}$$

Therefore, we obtain

$$\fint_{B_4^+} |Dh_l|^q dx \leq c. \tag{2.1.41}$$

Analogously, since $h_l = 0$ in T_4, it comes from the Poincaré inequality and (2.1.41) that

$$\fint_{B_4^+} |h_l|^q dx \leq c \fint_{B_4^+} |Dh_l|^q dx \leq c. \tag{2.1.42}$$

Therefore, in view of (2.1.38), (2.1.41) and (2.1.42), the claim (2.1.40) is valid, and so there exist a subsequence of $\{h_l\}_{l=1}^\infty$, which is still denoted by $\{h_l\}_{l=1}^\infty$, and a function $h_0 \in W^{2,q}(B_4^+)$ such that

$$\begin{cases} h_l \rightharpoonup h_0 & \text{weakly in } W^{2,q}(B_4^+), \\ h_l \to h_0 & \text{strongly in } L^q(B_4^+), \end{cases} \quad \text{as} \ \ l \to \infty. \tag{2.1.43}$$

In the same way that we have estimated (2.1.30), we have from (2.1.37) that

$$\mathbf{B}_l \longrightarrow \mathbf{B}_0 \ \text{ in } \ L^{q'}(B_4^+) \ \text{ as } \ l \to \infty \ \text{ (up to subsequence)}, \tag{2.1.44}$$

for some constant matrix $\mathbf{B}_0 = (b_{ij}^0)$. From (2.1.38), (2.1.43) and (2.1.44), we see that $h_0 \in W^{2,q}(B_4^+)$ is a solution of

$$\begin{cases} b_{ij}^0 D_{ij} h_0 = 0 & \text{in} \quad B_4^+, \\ h_0 = 0 & \text{on} \quad T_4. \end{cases}$$

In addition, it follows from (2.1.38) and (2.1.43) that

$$\fint_{B_4^+} |D^2 h_0|^q dx \leq \liminf_{l \to \infty} \fint_{B_4^+} |D^2 h_l|^q dx \leq 1. \tag{2.1.45}$$

Therefore, by considering the unique solution $v_l \in W^{2,q}(B_3^+)$ of

$$\begin{cases} \overline{b^l_{ij}}_{B_4^+} D_{ij} v_l &= 0 \quad \text{in} \quad B_3^+, \\ v_l &= h_0 \quad \text{on} \quad \partial B_3^+, \end{cases}$$

we extract a contradiction to the inequality (2.1.39) in the same way as in the proof of Lemma 2.1.5. This completes the proof. □

Corollary 2.1.8. *Under the hypotheses and conclusion of Lemma 2.1.7, we have*

$$⨍_{B_1^+} |D^2(w-v)|^q dx \le \epsilon.$$

Proof. It can be proved by the same argument as in the proof of Corollary 2.1.6. □

2.2 Weighted $W^{2,p}$-estimates

2.2.1 Preliminaries and main result

Before stating our main result of this chapter, Chapter 2.2, let us present some properties of the Muckenhoupt classes A_s, $1 < s < \infty$, which will be treated in this thesis. We say that w is a *weight* in *Muckenhoupt class* A_s, or an A_s *weight*, if w is a positive locally integrable function on $\mathbb{R}^n$ such that

$$[w]_s := \sup \left(⨍_B w(x)dx \right) \left(⨍_B w(x)^{\frac{-1}{s-1}} dx \right)^{s-1} < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. If w is an A_s weight, we write $w \in A_s$, and $[w]_s$ is called the A_s *constant of* w. The A_s class is stable with respect to translation, dilation and multiplication by a positive scalar. Every A_s weight has the doubling property, and the monotonicity $A_{s_1} \subset A_{s_2}$, $1 < s_1 \le s_2 < \infty$. A typical example of A_s weights for $1 < s < \infty$ is the function $w_\alpha(x) = |x|^\alpha$, $x \in \mathbb{R}^n$ where $-n < \alpha < n(s-1)$. We shall identify the weight w with the measure

$$w(E) = \int_E w dx,$$

for measurable sets $E \subset \mathbb{R}^n$.

Related to the A_s weight w is the *weighted Lebesgue space* $L^s_w(\Omega), 1 <$

$s < \infty$, which contains all measurable functions g on Ω such that

$$\|g\|_{L^s_w(\Omega)} := \left(\int_\Omega |g|^s w dx\right)^{1/s} < +\infty.$$

Given $w \in A_s$, $1 < s < \infty$ and a nonnegative integer m, we also define the *weighted Sobolev space* $W^{m,s}_w(\Omega)$ as the set of functions $g \in L^s_w(\Omega)$ with weak derivatives $D^\alpha g \in L^s_w(\Omega)$ for $|\alpha| \le m$. The norm of g in $W^{m,s}_w(\Omega)$ is given by

$$\|g\|_{W^{m,s}_w(\Omega)} := \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha g|^s \, w dx\right)^{\frac{1}{s}}.$$

The following is an important property of the A_s weights (see [70] for details).

Lemma 2.2.1. *Let w be an A_s weight for some $1 < s < \infty$, and let E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then there exist two constants $\beta, \nu > 0$ depending only on n and w such that*

$$[w]_s^{-1} \left(\frac{|E|}{|B|}\right)^s \le \frac{w(E)}{w(B)} \le \beta \left(\frac{|E|}{|B|}\right)^\nu.$$

Unless otherwise stated, we assume that w is an $A_{\frac{p}{2}}$ weight for $2 < p < \infty$ throughout the thesis. Let us now state one of the main theorems in this chapter.

Theorem 2.2.2 (Main Theorem)**.** *Given $2 < p < \infty$ and a weight $w \in A_{\frac{p}{2}}$, there exists a small $\delta = \delta(\Lambda, p, n, w, \partial\Omega) > 0$ so that if $\mathbf{A}$ is uniformly elliptic and (δ, R)-vanishing, $\partial\Omega \in C^{1,1}$ and $|f|^2 \in L^{\frac{p}{2}}_w(\Omega)$, then the solution $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ of (2.0.1) satisfies $|D^2u|^2 \in L^{\frac{p}{2}}_w(\Omega)$ with the estimate*

$$\int_\Omega |D^2 u|^p w dx \le c \int_\Omega |f|^p w dx,$$

where a constant $c > 0$ is independent of u and f.

A strong solution of the equation (2.0.1), which is treated throughout the thesis, is a twice weakly differentiable function satisfying the equation (2.0.1) almost everywhere in Ω and assuming boundary values on $\partial\Omega$ in classical or in general sense, while a classical solution of the equation must be at least twice continuously differentiable. Since $L^{\frac{p}{2}}_w(\Omega) \subset L^1(\Omega)$ for $2 < p < \infty$, we remark that there is a unique strong solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of the problem (2.0.1) under the given conditions including $|f|^2 \in L_w^{\frac{p}{2}}(\Omega)$ according to the results in [25].

One of the main tools in our approach for proving the main theorem is the Hardy-Littlewood maximal function which controls the local behavior of a function. For a locally integrable function g defined in $\mathbb{R}^n$, we denote the maximal function of g by

$$\mathcal{M}g(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| dy,$$

at each point $x \in \mathbb{R}^n$. We also use

$$\mathcal{M}_\Omega g = \mathcal{M}\left(\chi_\Omega g\right)$$

if g is not defined outside Ω.

We shall use the basic properties of the Hardy-Littlewood maximal function as follows:
(1) (strong $p-p$ estimate)

$$\|\mathcal{M}g\|_{L^p(\mathbb{R}^n)} \leq c\|g\|_{L^p(\mathbb{R}^n)} \quad \text{for } \ 1 < p \leq \infty,$$

where a constant c depends only on n and p.
(2) (weak $1-1$ estimate)

$$|\{x \in \mathbb{R}^n : \mathcal{M}g(x) \geq t\}| \leq \frac{c}{t}\|g\|_{L^1(\mathbb{R}^n)} \quad \text{for } \ \forall t > 0,$$

where a constant c depends only on n.

The following is the so-called Muckenhoupt's theorem (see [60] for details). Since $L_w^s(\mathbb{R}^n) \subset L_{loc}^1(\mathbb{R}^n)$ for $1 < s < \infty$, $\mathcal{M}g$ is meaningful when $g \in L_w^s(\mathbb{R}^n)$.

Lemma 2.2.3. *Suppose $w \in A_s$ where $1 < s < \infty$. Then there exists a constant $c = c(n, s, [w]_s) > 0$ such that*

$$\int_{\mathbb{R}^n} (\mathcal{M}g)^s \, w dx \leq c \int_{\mathbb{R}^n} |g|^s w dx \tag{2.2.1}$$

whenever $g \in L_w^s(\mathbb{R}^n)$. Conversely, if (2.2.1) holds for every $g \in L_w^s(\mathbb{R}^n)$, then $w \in A_s$.

We also need the following standard measure theory; see [17, 22].

Lemma 2.2.4. *Suppose g is a nonnegative measurable function in a bounded domain $\Omega \subset \mathbb{R}^n$. Let $\eta > 0$ and $M > 1$ be constants and w be a weight in $\mathbb{R}^n$. Then for $0 < s < \infty$,*

$$g \in L^s_w(\Omega) \text{ if and only if } S := \sum_{k\geq 1} M^{sk} w\left(\{x \in \Omega : g(x) > \eta M^k\}\right) < \infty$$

and moreover

$$c^{-1} S \leq \|g\|^s_{L^s_w(\Omega)} \leq c(w(\Omega) + S), \tag{2.2.2}$$

where $c > 0$ is a constant depending only on η, M and s.

Proof. A direct computation provides that

$$\begin{aligned}
&\int_\Omega |g|^q \, w \, dx \\
&= \int_{\{x\in\Omega\colon\, |g(x)|\leq \eta M\}} |g|^q \, w \, dx \; + \sum_{k\geq 1} \int_{\{x\in\Omega\colon\, \eta M^k < |g(x)| \leq \eta M^{k+1}\}} |g|^q \, w \, dx \\
&\leq |\eta M|^q \, w(\Omega) + \sum_{k\geq 1} |\eta M^{k+1}|^q \, w\left(\{x \in \Omega\colon |g(x)| > \eta M^k\}\right) \\
&= (\eta M)^q \, w(\Omega) + (\eta M)^q \sum_{k\geq 1} M^{qk} \, w\left(\{x \in \Omega\colon |g(x)| > \eta M^k\}\right) \\
&= (\eta M)^q \left(w(\Omega) + S\right),
\end{aligned}$$

which implies the second inequality of (2.2.2). On the other hand, by Fubini's theorem, we infer that

$$\begin{aligned}
\int_\Omega |g|^q \, w \, dx &= \int_\Omega w(x) \left(\int_0^{|g(x)|} d(\tau^q)\right) dx \\
&= \int_0^\infty \left(\int_{\{x\in\Omega\colon\, |g(x)|>\tau\}} w(x) dx\right) d(\tau^q) \\
&\geq \sum_{k\geq 1} \int_{\eta M^{k-1}}^{\eta M^k} \left(\int_{\{x\in\Omega\colon\, |g(x)|>\tau\}} w(x) dx\right) d(\tau^q) \\
&\geq \sum_{k\geq 1} \left(\int_{\{x\in\Omega\colon\, |g(x)|>\eta M^k\}} w(x) dx\right) \int_{\eta M^{k-1}}^{\eta M^k} d(\tau^q) \\
&= \sum_{k\geq 1} \left((\eta M^k)^q - (\eta M^{k-1})^q\right) w\left(\{x \in \Omega\colon |g(x)| > \eta M^k\}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \eta^q(1-M^{-q})\sum_{k\geq 1} M^{kq} w\left(\{x\in\Omega\colon |g(x)|>\eta M^k\}\right)\\
&= \eta^q(1-M^{-q})\,S,
\end{aligned}
$$

which means the first inequality of (2.2.2). Therefore, we finally get

$$\eta^q(1-M^{-q})\,S \leq \|g\|^q_{L^q_w(\Omega)} \leq (\eta M)^q\,(w(\Omega)+S)\,.$$

□

We next introduce one of main tools which will be used repeatedly in the proofs of the weighted interior and boundary $W^{2,p}$ estimates.

Lemma 2.2.5. *(Vitali Covering Lemma) Let $\mathcal{C}$ be a class of balls B_α in $\mathbb{R}^n$ with their radii bounded above. Then there exist disjoint balls $\{B_{\alpha_i}\}_{i=1}^\infty \subset \{B_\alpha\}_\alpha$ such that*

$$\bigcup_\alpha B_\alpha \subset \bigcup_i 5B_{\alpha_i},$$

where $5B_{\alpha_i}$ denotes the ball with the same center as B_{α_i} but with five times the radius.

Indeed, we shall employ the following modified versions of Vitali covering lemma. They can be obtained from the above Vitali covering lemma; see the papers [11] and [71] for their proofs and more details.

Lemma 2.2.6. *Let $0<\epsilon<1$ and E and F be measurable sets with $E\subset F\subset B_1$ such that*
(1) $|E|<\epsilon|B_1|$ and
(2) for every $x\in B_1$ with $|E\cap B_r(x)|\geq\epsilon|B_r|$, $B_r(x)\cap B_1\subset F$.
Then $|E|\leq 10^n\epsilon|F|$.

Lemma 2.2.7. *Let $0<\epsilon<1$, and E and F be measurable sets with $E\subset F\subset B_1^+$ such that*
(1) $|E|<\epsilon|B_1^+|$ and
(2) for every $x\in B_1^+$ with $|E\cap B_r(x)|\geq\epsilon|B_r|$, $B_r(x)\cap B_1^+\subset F$.
Then $|E|\leq 2(10^n)\epsilon|F|$.

The following lemma is the weighted version of the modified Vitali covering lemma.

Lemma 2.2.8. *Let w be an A_s weight for some $1<s<\infty$. Let $0<\epsilon<1$ and suppose that the measurable sets E and F with $E\subset F\subset B_1^+$ satisfy the*

following properties:
(1) $w(E) < \epsilon w(B_1^+)$, *and*
(2) for every $x \in B_1^+$ *and* $0 < r \leq 1$,

$$w(E \cap B_r(x)) \geq \epsilon w(B_r(x)) \text{ implies } B_r(x) \cap B_1^+ \subset F.$$

Then $w(E) \leq 20^{ns}\epsilon [w]_s^2 w(F)$.

Proof. In view of (1), for almost all $x \in E$, there is a small $\rho_x > 0$ such that

$$w(E \cap B_{\rho_x}(x)) = \epsilon w(B_{\rho_x}(x)) \text{ and } w(E \cap B_\rho(x)) < \epsilon w(B_\rho(x)) \tag{2.2.3}$$

for any $\rho \in (\rho_x, 1]$. Since $\{B_{\rho_x}(x)\}_{x \in E}$ covers E with $\rho_x \leq 1$, the Vitali covering lemma, Lemma 2.2.5, implies that there is a countable $\{x_i\}_{i=1}^\infty$ so that the balls $B_{\rho_{x_i}}(x_i)$ are mutually disjoint and $E \subset \bigcup_i B_{5\rho_{x_i}}(x_i)$. Then by Lemma 2.2.1 and (2.2.3),

$$w(E \cap B_{5\rho_{x_i}}(x_i)) < \epsilon w(B_{5\rho_{x_i}}(x_i)) \leq \epsilon [w]_s 5^{ns} w(B_{\rho_{x_i}}(x_i)).$$

We notice that

$$\sup_{0<\rho\leq 1} \sup_{x \in B_1^+} \frac{|B_\rho(x)|}{|B_\rho(x) \cap B_1^+|} \leq 4^n.$$

Therefore from Lemma 2.2.1, we finally obtain

$$\begin{aligned}
w(E) &\leq w\left(E \cap \bigcup_{i\geq 1} B_{5\rho_{x_i}}(x_i)\right) \leq \sum_{i \geq 1} \epsilon [w]_s 5^{ns} w(B_{\rho_{x_i}}(x_i)) \\
&\leq [w]_s^2 5^{ns} \epsilon \sum_{i\geq 1} \left(\frac{|B_{\rho_{x_i}}(x_i)|}{|B_{\rho_{x_i}}(x_i) \cap B_1^+|}\right)^s w(B_{\rho_{x_i}}(x_i) \cap B_1^+) \\
&\leq [w]_s^2 20^{ns} \epsilon \sum_{i \geq 1} w(B_{\rho_{x_i}}(x_i) \cap B_1^+) \\
&\leq [w]_s^2 20^{ns} \epsilon w\left(\bigcup_{i \geq 1} B_{\rho_{x_i}}(x_i) \cap B_1^+\right) \leq 20^{ns}\epsilon [w]_s^2 w(F),
\end{aligned}$$

where the last inequality comes from (2.2.3) and the second hypothesis. □

2.2.2 Interior weighted estimates

In this section, we shall prove the interior weighted $W^{2,p}$ estimates for the nondivergence type elliptic equation (2.0.1) via the so-called maximal function approach, which is different from those previously used, for instance, in [24, 53]. We begin with the interior unweighted $W^{2,2}$ estimates for the equation (2.0.1) from [24].

Lemma 2.2.9. *There exists a small $\delta = \delta(\Lambda, n) > 0$ such that if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing and if $f \in L^2(B_6)$, then for any solution $u \in W^{2,2}(B_6)$ of*

$$a_{ij} D_{ij} u = f \quad \text{in } B_6, \tag{2.2.4}$$

we have the estimate

$$\|D^2 u\|_{L^2(B_1)} \leq c \left(\|f\|_{L^2(B_6)} + \|u\|_{L^2(B_6)} \right),$$

where a constant $c > 0$ is independent of u and f.

The following is the main theorem in this section.

Theorem 2.2.10. *Given $2 < p < \infty$ and a weight $w \in A_{\frac{p}{2}}$, there exists a small $\delta = \delta(\Lambda, p, n, w) > 0$ such that if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing and if $|f|^2 \in L_w^{\frac{p}{2}}(B_6)$, then for any solution $u \in W^{2,2}(B_6)$ of (2.2.4), we have $|D^2 u|^2 \in L_w^{\frac{p}{2}}(B_1)$ with the estimate*

$$\|D^2 u\|_{L_w^p(B_1)} \leq c \left(\|f\|_{L_w^p(B_6)} + \|u\|_{L^2(B_6)} \right), \tag{2.2.5}$$

where a constant $c > 0$ is independent of u and f.

Lemma 2.2.11. *There is a positive constant $N_1 = N_1(\Lambda, n)$ so that for any $\epsilon > 0$ there exists a small $\delta = \delta(\epsilon, \Lambda, n) > 0$ such that if $u \in W^{2,2}(\Omega)$ is a solution of*

$$a_{ij} D_{ij} u = f \ \text{ in } \Omega \supset B_6 \tag{2.2.6}$$

with

$$\left\{ x \in \Omega : \mathcal{M}\left(|D^2 u|^2\right)(x) \leq 1 \right\} \cap \left\{ x \in \Omega : \mathcal{M}\left(|f|^2\right)(x) \leq \delta^2 \right\} \cap B_1 \neq \emptyset \tag{2.2.7}$$

and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing, then

$$\left| \left\{ x \in \Omega : \mathcal{M}\left(|D^2 u|^2\right)(x) > N_1^2 \right\} \cap B_1 \right| < \epsilon |B_1|.$$

Proof. From the condition (2.2.7), there is a point $x_0 \in B_1$ such that

$$\frac{1}{|B_\rho|} \int_{B_\rho(x_0) \cap \Omega} |D^2 u|^2 dx \leq 1 \quad \text{and} \quad \frac{1}{|B_\rho|} \int_{B_\rho(x_0) \cap \Omega} |f|^2 dx \leq \delta^2,$$

for any $\rho > 0$. Note $B_4 \subset B_5(x_0)$ to see that

$$\fint_{B_4} |D^2u|^2 dx \le \left(\frac{5}{4}\right)^n \fint_{B_5(x_0)\cap\Omega} |D^2u|^2 dx \le 2^n.$$

Likewise, we have that

$$\fint_{B_4} |f|^2 dx \le 2^n \delta^2.$$

Then we apply Corollary 2.1.2 to the equation (2.2.13) with u and f replaced by $(\frac{1}{2^{\frac{n}{2}}})u$ and $(\frac{1}{2^{\frac{n}{2}}})f$ respectively, in order to find that for any $\eta > 0$, there exist a small $\delta = \delta(\eta) > 0$, a positive constant $N_0 = N_0(n, \Lambda)$, a constant matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})$ with $\|\overline{\mathbf{A}}_{B_4} - \tilde{\mathbf{A}}\|_{L^\infty(\mathbb{R}^n)} \le \eta$ and a solution $v \in W^{2,2}(B_4)$ of

$$\tilde{a}_{ij} D_{ij} v = 0 \text{ in } B_4,$$

such that

$$\|D^2 v\|^2_{L^\infty(B_3)} \le N_0^2 \quad \text{and} \quad \fint_{B_2} |D^2(u-v)|^2 dx \le \eta^2,$$

provided that

$$\fint_{B_4} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4}|^2 dx \le \delta^2.$$

We next write $N_1 = \max\{4N_0^2, 2^n\}$ and claim

$$\begin{aligned} &\left\{x \in B_1 : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \\ &\quad \subset \left\{x \in B_1 : \mathcal{M}_{B_4}\left(|D^2(u-v)|^2\right)(x) > N_0^2\right\}. \end{aligned} \tag{2.2.8}$$

Indeed, suppose $x_1 \in \left\{x \in B_1 : \mathcal{M}_{B_4}\left(|D^2(u-v)|^2\right)(x) \le N_0^2\right\}$. Then for $\rho \le 2$, $B_\rho(x_1) \subset B_3$ and so

$$\begin{aligned} \fint_{B_\rho(x_1)} |D^2u|^2 dx &\le 2\fint_{B_\rho(x_1)} \left(|D^2(u-v)|^2 + |D^2v|^2\right) dx \\ &\le 2\mathcal{M}_{B_4}\left(|D^2(u-v)|^2\right)(x_1) + 2N_0^2 \\ &\le 4N_0^2. \end{aligned}$$

On the other hand, if $\rho > 2$, $x_0 \in B_\rho(x_1) \subset B_{2\rho}(x_0)$, and so we find that

$$\fint_{B_\rho(x_1)} |D^2u|^2 dx \le 2^n \fint_{B_{2\rho}(x_0)} |D^2u|^2 dx \le 2^n.$$

Therefore, $x_1 \in \left\{x \in B_1 : \mathcal{M}\left(|D^2u|^2\right)(x) \leq N_1^2\right\}$, and so the claim (2.2.8) is proved.

From (2.2.8) and the weak 1-1 estimate, we finally get

$$\begin{aligned}
&\frac{1}{|B_1|}\left|\left\{x \in B_1 : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\}\right| \\
&\leq \frac{1}{|B_1|}\left|\left\{x \in B_1 : \mathcal{M}_{B_4}\left(|D^2(u-v)|^2\right)(x) > N_0^2\right\}\right| \\
&\leq c \fint_{B_2} |D^2(u-v)|^2 dx \leq c\eta^2 < \epsilon
\end{aligned}$$

by taking η and δ satisfying the last inequality above, with c being depending only on n, Λ. □

With the above lemma, Lemma 2.2.11, we have its weighted version whose proof is similar to that of Lemma 2.2.18.

Lemma 2.2.12. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$, $y \in \Omega$ and $r > 0$. Then there is a constant $N_1(n, \Lambda) > 0$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon, \Lambda, n, w, s) > 0$ such that if $u \in W^{2,2}(\Omega)$ is a solution of $a_{ij}D_{ij}u = f$ in $\Omega \supset B_{6r}(y)$ with*

$$\left\{x \in \Omega : \mathcal{M}\left(|D^2u|^2\right)(x) \leq 1\right\} \cap \left\{x \in \Omega : \mathcal{M}\left(|f|^2\right)(x) \leq \delta^2\right\} \cap B_r(y) \neq \emptyset$$

and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6r)$-vanishing, then we have

$$w\left(\left\{x \in B_1 : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \cap B_r(y)\right) < \epsilon w\left(B_r(y)\right).$$

By a scaling argument, we now have the following lemma.

Lemma 2.2.13. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$. Then there is a constant $N_1 = N_1(n, \Lambda) > 0$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon, \Lambda, n, w, s) > 0$ such that if $u \in W^{2,2}(\Omega)$ is a solution of $a_{ij}D_{ij}u = f$ in $\Omega \supset B_6$ with*

$$w\left(\left\{x \in B_1 : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \cap B_r(y)\right) \geq \epsilon w\left(B_r(y)\right)$$

for all $y \in B_1$ and for all $r \in (0, \frac{1}{2})$, and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing, then we have

$$B_r(y) \cap B_1 \subset \left\{x \in B_1 : \mathcal{M}\left(|D^2u|^2\right)(x) > 1\right\} \cup \left\{x \in B_1 : \mathcal{M}\left(|f|^2\right)(x) > \delta^2\right\}.$$

In view of Lemma 2.2.6, we derive the following power decay estimate. We refer to the proof of Lemma 2.2.20 for its completeness.

Lemma 2.2.14. *Under the same assumptions as in Lemma 2.2.13, we further assume*

$$w\left(\left\{x \in \Omega : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \cap B_1\right) < \epsilon w\left(B_1\right).$$

Then we have

$$\begin{aligned} & w\left(\left\{x \in B_1 : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^{2k}\right\}\right) \\ & \leq \epsilon_1^k w\left(\left\{x \in B_1 : \mathcal{M}\left(|D^2u|^2\right)(x) > 1\right\}\right) \\ & \quad + \sum_{i=1}^{k} \epsilon_1^i w\left(\left\{x \in B_1 : \mathcal{M}\left(|f|^2\right)(x) > \delta^2 N_1^{2(k-i)}\right\}\right), \end{aligned}$$

where $\epsilon_1 = 10^{ns}\epsilon[w]_s^2$.

Lemma 2.2.12, Lemma 2.2.13 and Lemma 2.2.14 can be proved in an analogous way to Lemma 2.2.18, Lemma 2.2.19 and Lemma 2.2.20, respectively, and so we omit their proofs.

We are now ready to prove the main theorem of this section. Let us take N_1, ϵ and the corresponding δ to be the same as in the previous lemma.

Proof of Theorem 2.2.10. In this proof, we denote c to mean a universal constant which can be computed in terms of n, Λ, p and w. From the assumptions that $|f|^2 \in L_w^{\frac{p}{2}}(B_6)$ and $w \in A_{\frac{p}{2}}$, we find

$$\int_{B_6} |f|^2 dx \leq \left(\int_{B_6} |f|^p w dx\right)^{\frac{2}{p}} \left(w^{\frac{-2}{p-2}}(B_6)\right)^{\frac{p-2}{p}} \leq c\|f\|^2_{L^p_w(B_6)}, \tag{2.2.9}$$

by using Hölder inequality, and so it turns out that $|f| \in L^2(B_6)$. Then by Lemma 2.2.9, there exists a unique solution u of (2.2.4) with the estimate

$$\|D^2u\|_{L^2(B_1)} \leq c\left(\|f\|_{L^2(B_6)} + \|u\|_{L^2(B_6)}\right). \tag{2.2.10}$$

Considering

$$\tilde{u} = \frac{\delta u}{\left(\|f\|_{L^p_w(B_6)} + \|u\|_{L^2(B_6)}\right)}$$

and

$$\tilde{f} = \frac{\delta f}{\left(\|f\|_{L^p_w(B_6)} + \|u\|_{L^2(B_6)}\right)},$$

we observe that $\tilde{u} \in W^{2,2}(B_6)$ is a solution of

$$a_{ij} D_{ij} \tilde{u} = \tilde{f} \quad \text{in } B_6$$

with $\|\tilde{f}\|_{L^2(B_6)} + \|\tilde{u}\|_{L^2(B_6)} \leq c\|\tilde{f}\|_{L^p_w(B_6)} + \|\tilde{u}\|_{L^2(B_6)} \leq c\delta$. Then it follows from (2.2.9), (2.2.10) and the weak 1-1 estimate that

$$\begin{aligned}
&\frac{1}{|B_1|} \left|\left\{x \in B_1 : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > N_1^2\right\}\right| \\
&\quad \leq c \fint_{B_1} |D^2\tilde{u}|^2 dx \leq c\left(\fint_{B_6} |\tilde{f}|^2 dx + \fint_{B_6} |\tilde{u}|^2 \, dx\right) \leq c\delta^2.
\end{aligned}$$

We then recall Lemma 2.2.1 to discover that

$$\begin{aligned}
&\frac{1}{w(B_1)} w\left(\left\{x \in B_1 : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > N_1^2\right\}\right) \\
&\quad \leq \beta \left(\frac{\left|\left\{x \in B_1 : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > N_1^2\right\}\right|}{|B_1|}\right)^{\nu} \leq c\beta\delta^{2\nu} < \epsilon,
\end{aligned}$$

by taking δ in order to get the last inequality. Thus all the hypotheses of Lemma 2.2.14 are satisfied. We recall Lemma 2.2.3 and Lemma 2.2.4 to observe that

$$\sum_{k=1}^{\infty} N_1^{pk} w\left(\{x \in B_1 : \mathcal{M}\left(|\tilde{f}|^2\right)(x) > \delta^2 N_1^{2k}\}\right) \leq c \left\|\frac{\tilde{f}}{\delta}\right\|^p_{L^p_w(B_6)} \leq c.$$

Then by Lemma 2.2.14, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} N_1^{pk} w\left(\left\{x \in B_1 : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > N_1^{2k}\right\}\right) \\
&\leq \sum_{k=1}^{\infty} N_1^{pk} \Bigg\{ \epsilon_1^k w\left(\{x \in B_1 : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > 1\}\right) \\
&\qquad\qquad + \sum_{i=1}^{k} \epsilon_1^i w\left(\left\{x \in B_1 : \mathcal{M}\left(|\tilde{f}|^2\right)(x) > \delta^2 N_1^{2(k-i)}\right\}\right) \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} N_1^{pk} \epsilon_1^k w\left(\left\{x \in B_1 : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > 1\right\}\right) \\
&\quad + \sum_{i=1}^{\infty} (N_1^p \epsilon_1)^i \left(\sum_{k=i}^{\infty} N_1^{p(k-i)} w\left(\left\{x \in B_1 : \mathcal{M}\left(|\tilde{f}|^2\right)(x) > \delta^2 N_1^{2(k-i)}\right\}\right)\right) \\
&\leq \sum_{k=1}^{\infty} \left(N_1^p \epsilon_1\right)^k (w(B_1) + c)\,.
\end{aligned}
$$

We now take ϵ_1 so that $N_1^p \epsilon_1 < 1$, and then conclude from Lemma 2.2.3 and Lemma 2.2.4 that $\|D^2\tilde{u}\|_{L^p_w(B_1)} \leq c^*$ for some positive constant $c^* = c^*(\Lambda, n, p, w)$. We return from $\tilde{u}$ to u and make a standard procedure for higher integrability for u, to finally derive the desired estimate (2.2.5). $\square$

2.2.3 Boundary weighted estimates

In this section, we derive a weighted $W^{2,p}$ estimate on the flat boundary. To this end, we consider a special case that the domain under consideration is a half ball. We start with an unweighted $W^{2,2}$ estimate near the flat boundary from [25].

Lemma 2.2.15. *There exists a small $\delta = \delta(\Lambda, n) > 0$ so that if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing and if $f \in L^2(B_6^+)$, then for any solution $u \in W^{2,2}(B_6^+)$ of*

$$
\begin{cases} a_{ij} D_{ij} u = f & \text{in} \quad B_6^+, \\ u = 0 & \text{on} \quad T_6, \end{cases}
$$

we have the estimate

$$
\|D^2 u\|_{L^2(B_1^+)} \leq c \left(\|f\|_{L^2(B_6^+)} + \|u\|_{L^2(B_6^+)} \right)
$$

where a constant $c > 0$ is independent of u and f.

We now state the main theorem in this section.

Theorem 2.2.16. *Given $2 < p < \infty$ and a weight $w \in A_{\frac{p}{2}}$, there exists a small $\delta = \delta(\Lambda, p, n, w) > 0$ such that if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing and if $|f|^2 \in L_w^{\frac{p}{2}}(B_6^+)$, then for any solution $u \in W^{2,2}(B_6^+)$ of*

$$
\begin{cases} a_{ij} D_{ij} u = f & \text{in} \quad B_6^+, \\ u = 0 & \text{on} \quad T_6, \end{cases} \tag{2.2.11}
$$

we have $|D^2u|^2 \in L_w^{\frac{p}{2}}(B_1^+)$ *with the estimate*

$$\|D^2u\|_{L_w^p(B_1^+)} \leq c\left(\|f\|_{L_w^p(B_6^+)} + \|u\|_{L^2(B_6^+)}\right) \tag{2.2.12}$$

where a constant $c > 0$ *is independent of* u *and* f.

Lemma 2.2.17. *There is a positive constant* $N_1 = N_1(n, \Lambda)$ *so that for any* $\epsilon > 0$, *there exists a small* $\delta = \delta(\epsilon, \Lambda, n) > 0$ *such that if* $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ *is a solution of*

$$\begin{cases} a_{ij}D_{ij}u &= f \quad \text{in} \quad \Omega \supset B_6^+, \\ u &= 0 \quad \text{on} \quad \partial\Omega \supset T_6, \end{cases} \tag{2.2.13}$$

with

$$B_1^+ \cap \left\{x \in \Omega : \mathcal{M}\left(|D^2u|^2\right)(x) \leq 1\right\} \cap \left\{x \in \Omega : \mathcal{M}\left(|f|^2\right)(x) \leq \delta^2\right\} \neq \emptyset, \tag{2.2.14}$$

and if $\mathbf{A}$ *is uniformly elliptic and* $(\delta, 6)$*-vanishing, then*

$$|\left\{x \in \Omega : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \cap B_1^+| < \epsilon|B_1^+|.$$

Proof. From the hypothesis (2.2.14), there exists a point $x_0 \in B_1^+$ so that

$$\frac{1}{|B_\rho|}\int_{B_\rho^+(x_0)\cap\Omega} |D^2u|^2dx \leq 1 \text{ and } \frac{1}{|B_\rho|}\int_{B_\rho^+(x_0)\cap\Omega} |f|^2dx \leq \delta^2 \quad \text{for all } \rho > 0.$$

Since $B_4^+ \subset B_5^+(x_0)$, we can get

$$\fint_{B_4^+} |D^2u|^2dx \leq \left(\frac{5}{4}\right)^n \fint_{B_5^+(x_0)} |D^2u|^2dx \leq 2^n$$

and similarly

$$\fint_{B_4^+} |f|^2dx \leq 2^n\delta^2.$$

Let us apply Corollary 2.1.4 to the equation (2.2.13) with u and f replaced by $(\frac{1}{2^{\frac{n}{2}}})u$ and $(\frac{1}{2^{\frac{n}{2}}})f$ respectively, in order to have that for any $\eta > 0$, there exist a small $\delta = \delta(\eta) > 0$, a positive constant $N_0 = N_0(n, \Lambda)$, a constant matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})$ with $\left\|\overline{\mathbf{A}}_{B_4^+} - \tilde{\mathbf{A}}\right\|_{L^\infty(\mathbb{R}^n)} \leq \epsilon$ and a solution $v \in W^{2,2}(B_4^+)$ of

$$\begin{cases} \tilde{a}_{ij}D_{ij}v &= 0 \quad \text{in} \quad B_4^+, \\ v &= 0 \quad \text{on} \quad T_4 \end{cases}$$

such that

$$\|D^2 v\|^2_{L^\infty(B_3^+)} \le N_0^2 \text{ and } \fint_{B_2^+} |D^2(u-v)|^2 dx \le \eta^2,$$

provided that

$$\fint_{B_4^+} |f|^2 + |\mathbf{A} - \overline{\mathbf{A}}_{B_4^+}|^2 dx \le \delta^2.$$

Then we can now show in almost the same way as we did in the proof of Lemma 2.2.11 that

$$\left\{x \in B_1^+ : \mathcal{M}\left(|D^2 u|^2\right) > N_1^2\right\} \subset \left\{x \in B_1^+ : \mathcal{M}_{B_4^+}(|D^2(u-v)|^2) > N_0^2\right\}, \tag{2.2.15}$$

where $N_1^2 := \max\left\{4N_0^2, 2^n\right\}$. So we discover that for some $c = c(n, \Lambda) > 0$,

$$\begin{aligned}
&\frac{1}{|B_1^+|}\left|\left\{x \in B_1^+ : \mathcal{M}\left(|D^2 u|^2\right) > N_1^2\right\}\right| \\
&\le \frac{1}{|B_1^+|}\left|\left\{x \in B_1^+ : \mathcal{M}_{B_4^+}\left(|D^2(u-v)|^2\right) > N_0^2\right\}\right| \\
&\le c\fint_{B_2^+} |D^2(u-v)|^2 dx \le c\eta^2 < \epsilon,
\end{aligned}$$

if we take η and δ satisfying the last inequality above. □

Lemma 2.2.18. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$. There is a positive constant $N_1 = N_1(\Lambda, n)$ so that for any $\epsilon > 0$ and for every $0 < r \le 1$, there exists a small $\delta = \delta(\epsilon, \Lambda, n, w, s) > 0$ such that if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of*

$$\begin{cases} a_{ij} D_{ij} u &= f \quad \text{in} \quad \Omega \supset B_{6r}^+, \\ u &= 0 \quad \text{on} \quad \partial\Omega \supset T_{6r} \end{cases} \quad \text{with}$$

$$B_r^+ \cap \left\{x \in \Omega : \mathcal{M}(|D^2 u|^2)(x) \le 1\right\} \cap \left\{x \in \Omega : \mathcal{M}\left(|f|^2\right)(x) \le \delta^2\right\} \neq \emptyset, \tag{2.2.16}$$

and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6r)$-vanishing, then

$$w(\left\{x \in \Omega : \mathcal{M}\left(|D^2 u|^2\right)(x) > N_1^2\right\} \cap B_r^+) < \epsilon w(B_r^+).$$

Proof. Let us first define $\tilde{a}_{ij}(x) = a_{ij}(rx)$, $\tilde{u}(x) = \frac{1}{r^2}u(rx)$, $\tilde{f}(x) = f(rx)$ and $\tilde{\Omega} = \{\frac{1}{r}x : x \in \Omega\}$. Then we note that $\tilde{u} \in W^{2,2}(\tilde{\Omega}) \cap W_0^{1,2}(\tilde{\Omega})$ is the

solution of

$$\begin{cases} \tilde{a}_{ij} D_{ij} \tilde{u} &= \tilde{f} \quad \text{in} \quad \tilde{\Omega} \supset B_6^+, \\ \tilde{u} &= 0 \quad \text{on} \quad \partial\tilde{\Omega} \supset T_6. \end{cases}$$

Let $\epsilon > 0$ be given and choose $\delta = \delta(\epsilon, \Lambda, n, w, s)$ as in Lemma 2.2.17 with ϵ replaced by $(\frac{\epsilon}{2\beta})^{\frac{1}{\nu}}$, where β and ν are the constants in Lemma 2.2.1. From (2.2.16), there exists

$$x_0 \in B_r^+ \cap \left\{x \in \Omega : \mathcal{M}\left(|D^2u|^2\right)(x) \le 1\right\} \cap \left\{x \in \Omega : \mathcal{M}\left(|f|^2\right)(x) \le \delta^2\right\}.$$

Then we see

$$z_0 = \frac{x_0}{r} \in B_1^+ \cap \left\{z \in \tilde{\Omega} : \mathcal{M}(|D^2\tilde{u}|^2)(z) \le 1\right\} \cap \left\{z \in \tilde{\Omega} : \mathcal{M}\left(|\tilde{f}|^2\right)(z) \le \delta^2\right\}.$$

Since all the hypotheses of Lemma 2.2.17 are satisfied, Lemma 2.2.17 gives

$$\left|\{z \in \tilde{\Omega} : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(z) > N_1^2\} \cap B_1^+\right| < \left(\frac{\epsilon}{2\beta}\right)^{1/\nu} |B_1^+|.$$

Then

$$\left|\{x \in \Omega : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\} \cap B_r^+\right| < \left(\frac{\epsilon}{2\beta}\right)^{1/\nu} |B_r^+|. \tag{2.2.17}$$

Using Lemma 2.2.1, we finally get from (2.2.17) that

$$\begin{aligned} &w\left(\{x \in \Omega : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\} \cap B_r^+\right) \\ &\le \beta \left(\frac{\left|\{x \in \Omega : \mathcal{M}(|D^2u|^2)(x) > N_1^2\} \cap B_r^+\right|}{|B_r^+|}\right)^{\nu} w(B_r^+) < \epsilon w(B_r^+). \end{aligned}$$

□

Lemma 2.2.19. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$. Then there is a constant $N_1 = N_1(\Lambda, n, w) > 0$ so that for any $\epsilon > 0$, $0 < r \le \frac{1}{18}$ and $y \in B_1^+$, there exists a small $\delta = \delta(\epsilon, n, \Lambda, w) > 0$ such that if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of*

$$\begin{cases} a_{ij} D_{ij} u &= f \quad \textit{in} \quad \Omega \supset B_6^+, \\ u &= 0 \quad \textit{on} \quad \partial\Omega \supset T_6, \end{cases} \quad \textit{with}$$

$$w\left(\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\} \cap B_r(y)\right) \ge \epsilon w(B_r(y)) \tag{2.2.18}$$

and if $\mathbf{A}$ *is uniformly elliptic and* $(\delta, 6)$*-vanishing, then*

$$B_r(y) \cap B_1^+ \subset \left\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > 1\right\} \\ \cup \left\{x \in B_1^+ : \mathcal{M}\left(|f|^2\right)(x) > \delta^2\right\}. \quad (2.2.19)$$

Proof. We prove it by contradiction. To do this, assume that (2.2.18) holds and the conclusion (2.2.19) is false. Then there is a point $x_0 = ({x_0}', x_{0n}) \in B_r(y) \cap B_1^+$ such that

$$\frac{1}{|B_\rho|}\int_{B_\rho^+(x_0)\cap\Omega} |D^2u|^2 dx \le 1 \quad \text{and} \quad \frac{1}{|B_\rho|}\int_{B_\rho^+(x_0)\cap\Omega} |f|^2 dx \le \delta^2$$

for any $\rho > 0$. If $B_{6r}(x_0) \subset B_6^+$, it can be done from Lemma 2.2.13. Thus we need only to consider the case $B_{6r}(x_0) \not\subset B_6^+$, which implies $B_{6r}(x_0) \cap T_6 \neq \emptyset$. One can easily check that $({x_0}', 0) \in T_1$ and moreover

$$B_r(y) \cap B_1^+ \subset B_{6r}^+(x_0) \subset B_{12r}^+({x_0}', 0) \subset B_{72r}^+({x_0}', 0) \subset B_6^+ \subset \Omega,$$

for $0 < r \le \frac{1}{18}$. Apply Lemma 2.2.18 to $B_{12r}^+({x_0}', 0)$ with ϵ replaced by $\frac{2^\nu \epsilon}{\beta[w]_s\left(\frac{3}{5}\right)^{n\nu} 20^{ns}}$, to derive that

$$\begin{aligned} &\frac{1}{w\left(B_r(y)\right)} w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \cap B_r(y)\right) \\ &\le \frac{1}{w\left(B_r(y)\right)} w\left(\left\{x \in B_{12r}^+({x_0}', 0) : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\}\right) \\ &< \frac{2^\nu \epsilon w\left(B_{12r}^+({x_0}', 0)\right)}{\beta[w]_s\left(\frac{3}{5}\right)^{n\nu} 20^{ns} w\left(B_r(y)\right)}. \end{aligned}$$

However, Lemma 2.2.1 gives

$$\begin{aligned} &w\left(B_{12r}^+({x_0}', 0)\right) \\ &\le \beta\left(\frac{|B_{12r}^+|}{|B_{20r}|}\right)^\nu w\left(B_{20r}(y)\right) = \beta 2^{-\nu}\left(\frac{3}{5}\right)^{n\nu} w\left(B_{20r}(y)\right) \\ &\le \beta 2^{-\nu}\left(\frac{3}{5}\right)^{n\nu}[w]_s\left(\frac{|B_{20r}(y)|}{|B_r(y)|}\right)^s w(B_r(y)) \\ &= \beta 2^{-\nu}\left(\frac{3}{5}\right)^{n\nu}[w]_s 20^{ns} w(B_r(y)), \end{aligned}$$

since $B^+_{12r}(x_0{}', 0) \subset B_{20r}(y)$. Hence, we eventually obtain

$$w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \cap B_r(y)\right) < \epsilon w\left(B_r(y)\right),$$

which is a contradiction. □

Lemma 2.2.20. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$ and let N_1 be given by Lemma 2.2.19. For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, \Lambda, n, w, s) > 0$ such that if $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of*

$$\begin{cases} a_{ij}D_{ij}u = f & \text{in} \quad \Omega \supset B_6^+, \\ u = 0 & \text{on} \quad \partial\Omega \supset T_6, \end{cases} \quad \text{with}$$

$$w\left(\left\{x \in \Omega : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \cap B_1^+\right) < \epsilon w(B_1^+) \tag{2.2.20}$$

and if $\mathbf{A}$ is uniformly elliptic and $(\delta, 6)$-vanishing, then

$$\begin{aligned} & w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^{2k}\right\}\right) \\ & \le \epsilon_1^k w\left(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > 1\}\right) \\ & \quad + \sum_{i=1}^{k} \epsilon_1^i w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|f|^2\right)(x) > \delta^2 N_1^{2(k-i)}\right\}\right), \end{aligned}$$

where $\epsilon_1 := 20^{ns}\epsilon[w]_s^2$.

Proof. We use Lemma 2.2.8 on

$$\begin{aligned} E &:= \left\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\} \quad \text{and} \\ F &:= \left\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > 1\right\} \cup \left\{x \in B_1^+ : \mathcal{M}(|f|^2)(x) > \delta^2\right\}. \end{aligned}$$

From (2.2.20) and Lemma 2.2.19, we easily check that E and F satisfy the hypotheses of Lemma 2.2.8. Then Lemma 2.2.8 gives $w(E) \le \epsilon_1 w(F)$ with $\epsilon_1 := 20^{ns}\epsilon[w]_s^2$, that is,

$$\begin{aligned} & w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^2\right\}\right) \\ & \le \epsilon_1 w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > 1\right\}\right) \\ & \quad + \epsilon_1 w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|f|^2\right)(x) > \delta^2\right\}\right). \end{aligned}$$

For any $k \ge 2$, we know

$$E_k := \left\{x \in B_1^+ : \mathcal{M}\left(|D^2u|^2\right)(x) > N_1^k\right\} \subset E,$$

and so $w(E_k) < \epsilon w(B_1^+)$. Therefore for each $\lambda := N_1^{k-1}$, $u_\lambda := \frac{u}{\lambda} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is a solution of

$$\begin{cases} a_{ij}D_{ij}u_\lambda &= f_\lambda \quad \text{in} \quad \Omega \supset B_6^+, \\ u_\lambda &= 0 \quad \text{on} \quad \partial\Omega \supset T_6, \end{cases}$$

with $w(E_k^\lambda) < \epsilon w(B_1^+)$, and so

$$\begin{aligned} & w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|D^2u_\lambda|^2\right)(x) > N_1^2\right\}\right) \\ & \leq \epsilon_1 w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|D^2u_\lambda|^2\right)(x) > 1\right\}\right) \\ & \quad + \epsilon_1 w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|f_\lambda|^2\right)(x) > \delta^2\right\}\right), \end{aligned}$$

where $f_\lambda := \frac{f}{\lambda}$ and $E_k^\lambda := \left\{x \in B_1^+ : \mathcal{M}\left(|D^2u_\lambda|^2\right)(x) > N_1^k\right\}$. Hence we find

$$\begin{aligned} & w\left(\left\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^2\lambda^2\right\}\right) \\ & \leq \epsilon_1 w\left(\left\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > \lambda^2\right\}\right) \\ & \quad + \epsilon_1 w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|f|^2\right)(x) > \delta^2\lambda^2\right\}\right). \end{aligned}$$

Iterating the foregoing estimate, we finally derive

$$\begin{aligned} & w\left(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > N_1^{2k}\}\right) \\ & \leq \epsilon_1^k w\left(\{x \in B_1^+ : \mathcal{M}(|D^2u|^2)(x) > 1\}\right) \\ & \quad + \sum_{i=1}^{k} \epsilon_1^i w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|f|^2\right)(x) > \delta^2 N_1^{2(k-i)}\right\}\right), \end{aligned}$$

for any positive integer k. $\square$

Now we are ready to give a proof of Theorem 2.2.16. Let us take N_1, ϵ and the corresponding δ given by the previous lemma. Hereafter we employ c to denote any constant that can be computed in terms of n, Λ, p and w.

Proof of Theorem 2.2.16. Since $|f|^2 \in L_w^{\frac{p}{2}}(\Omega)$, we have

$$\int_\Omega |f|^2 dx \leq \left(\int_\Omega |f|^p w dx\right)^{\frac{2}{p}} \left(w^{\frac{-2}{p-2}}(\Omega)\right)^{\frac{p-2}{p}} \tag{2.2.21}$$

by Hölder inequality and so $|f| \in L^2(\Omega)$. Then Lemma 2.2.15 gives that there is a unique solution $u \in W^{2,2}(B_6^+)$ of (2.2.11) with the estimate

$$\|D^2 u\|_{L^2(B_1^+)} \leq c\left(\|f\|_{L^2(B_6^+)} + \|u\|_{L^2(B_6^+)}\right), \tag{2.2.22}$$

with the constant c independent of u and f.

We write $\tilde{u} = \frac{\delta u}{\left(\|f\|_{L^p_w(B_6^+)}+\|u\|_{L^2(B_6^+)}\right)}$ and $\tilde{f} = \frac{\delta f}{\left(\|f\|_{L^p_w(B_6^+)}+\|u\|_{L^2(B_6^+)}\right)}$. Then we see that

$$\|\tilde{f}\|_{L^2(B_6^+)} + \|\tilde{u}\|_{L^2(B_6^+)} \leq c\delta,$$

and $\tilde{u} \in W^{2,2}(B_6^+)$ is a solution of

$$\begin{cases} a_{ij}D_{ij}\tilde{u} &= \tilde{f} \quad \text{in} \quad B_6^+, \\ \tilde{u} &= 0 \quad \text{on} \quad T_6. \end{cases}$$

Then by (2.2.21), (2.2.22) and the weak 1-1 estimate, we deduce

$$\begin{aligned} &\frac{1}{|B_1^+|}\left|\left\{x \in B_1^+ : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > N_1^2\right\}\right| \leq c \fint_{B_1^+} |D^2\tilde{u}|^2 dx \\ &\leq c\left(\fint_{B_6^+} |\tilde{f}|^2 dx + \fint_{B_6^+} |\tilde{u}|^2 dx\right) \leq c\delta^2 \leq \left(\frac{\epsilon}{2\beta}\right)^{\frac{1}{\nu}}, \end{aligned}$$

by taking δ in order to get the last inequality, and hence Lemma 2.2.1 yields

$$\begin{aligned} &w\left(\left\{x \in B_1^+ : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > N_1^2\right\}\right) \\ &< \beta\left(\frac{\left|\left\{x \in B_1^+ : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > N_1^2\right\}\right|}{|B_1^+|}\right)^{\nu} w\left(B_1^+\right) < \epsilon w\left(B_1^+\right). \end{aligned}$$

We observe $|\tilde{f}|^2 \in L_w^{\frac{p}{2}}(B_6^+)$ with $\|\tilde{f}\|_{L^p_w(B_6^+)} \leq \delta$ and recall Lemmas 2.2.3 and 2.2.4 to discover that $\left\|\mathcal{M}\left(|\tilde{f}|^2\right)\right\|^{\frac{p}{2}}_{L_w^{\frac{p}{2}}(B_6^+)} \leq c\delta^p$, and so

$$\begin{aligned} &\sum_{k=1}^{\infty} N_1^{pk} w\left(\{x \in B_1^+ : \mathcal{M}\left(|\tilde{f}|^2\right)(x) > \delta^2 N_1^{2k}\}\right) \\ &\leq \sum_{k=1}^{\infty} N_1^{pk} w\left(\left\{x \in B_1^+ : \mathcal{M}\left(\left|\frac{\tilde{f}}{\delta}\right|^2\right)(x) > N_1^{2k}\right\}\right) \end{aligned}$$

$$\leq c \left\| \mathcal{M}\left(\left| \frac{\tilde{f}}{\delta} \right|^2 \right) \right\|_{L_w^{\frac{p}{2}}(B_6^+)}^{\frac{p}{2}} \leq c \left\| \frac{\tilde{f}}{\delta} \right\|_{L_w^p(B_6^+)}^p \leq c.$$

Therefore it follows from Lemma 2.2.20 that for some $c = c(n, \Lambda, w, p) > 0$,

$$\begin{aligned}
&\sum_{k=1}^{\infty} N_1^{pk} w\left(\{x \in B_1^+ : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > N_1^{2k}\} \right) \\
&\leq \sum_{k=1}^{\infty} N_1^{pk} \Bigg\{ \epsilon_1^k w\left(\{x \in B_1^+ : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > 1\}\right) \\
&\qquad\qquad + \sum_{i=1}^{k} \epsilon_1^i w\left(\{x \in B_1^+ : \mathcal{M}\left(|\tilde{f}|^2\right)(x) > \delta^2 N_1^{2(k-i)}\} \right) \Bigg\} \\
&= \sum_{k=1}^{\infty} N_1^{pk} \epsilon_1^k w\left(\{x \in B_1^+ : \mathcal{M}\left(|D^2\tilde{u}|^2\right)(x) > 1\}\right) \\
&\quad + \sum_{i=1}^{\infty} (N_1^p \epsilon_1)^i \left(\sum_{k=i}^{\infty} N_1^{p(k-i)} w\left(\{x \in B_1^+ : \mathcal{M}\left(|\tilde{f}|^2\right)(x) > \delta^2 N_1^{2(k-i)}\} \right) \right) \\
&\leq \sum_{k=1}^{\infty} N_1^{pk} \epsilon_1^k \left(w(B_1^+) + c \right) \leq c,
\end{aligned}$$

by taking ϵ_1 so that $N_1^p \epsilon_1 < 1$. Then we employ once again Lemmas 2.2.3 and 2.2.4 to find $\left\| D^2\tilde{u} \right\|_{L_w^p(B_1^+)} \leq c$, which in turn implies the desired estimate (2.2.12). $\square$

2.2.4 Global weighted estimates

In this section, we shall prove our main result, Theorem 2.2.2, via standard covering and flattening arguments. To be brief, we first derive the a priori weighted $W^{2,p}$ estimate from the interior and boundary estimates which we have obtained in the previous sections. We then remove the a priori assumption by an approximation procedure, to complete our proof. Once again we denote by c to mean a universal constant being dependent only on n, Λ, w, p and $\partial\Omega$.

Proof of Theorem 2.2.2. We start with the a priori assumption that

$$u \in W_w^{2,p}(\Omega). \tag{2.2.23}$$

Fix any point $x_0 \in \partial\Omega$. Since $\partial\Omega \in C^{1,1}$, we assume that

$$\Omega \cap B_r(x_0) = \{x \in \Omega : x_n > \gamma(x')\} \cap B_r(x_0)$$

for some small $r > 0$ and for some $C^{1,1}$ function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\frac{\partial\gamma}{\partial x_i}(x_0') = 0$ for any $i = 1, 2, \cdots, n-1$ and $\left\|\bigtriangledown^2\gamma\right\|_{L^\infty(\mathbb{R}^{n-1})} < \infty$.

We now use change variables to flatten out the boundary near x_0. To do this, define

$$\begin{cases} y_i &= x_i &=: \Phi^i(x), \quad \text{if } i = 1, 2, \cdots, n-1, \\ y_n &= x_n - \gamma(x') &=: \Phi^n(x), \end{cases}$$

and write $y = \Phi(x)$. We set $\Phi := \Psi^{-1}$ and write $x = \Psi(y)$. Choose $s > 0$ so small that the half ball $B_{12s}^+ \subset \Phi(\Omega \cap B_r(x_0))$. Define $\tilde{u}(y) = u(\Psi(y)) = u(x)$ for $y \in B_{6s}^+$ and $\tilde{w}(y) = w(\Psi(y))$ for $y \in \mathbb{R}^n$. Then it can be readily checked that $\tilde{w} \in A_{\frac{p}{2}}$ and $\tilde{u} \in W^{2,2}(B_{6s}^+)$ is a solution of

$$\begin{cases} \tilde{a}_{lm} D_{y_l y_m}\tilde{u} &= \tilde{f} \quad \text{in} \quad B_{6s}^+, \\ \tilde{u} &= 0 \quad \text{on} \quad T_{6s}, \end{cases}$$

where

$$\begin{aligned} \tilde{a}_{lm}(y) &= a_{ij}(\Psi(y))\Phi^l_{x_i}(\Psi(y))\Phi^m_{x_j}(\Psi(y)), \text{ and} \\ \tilde{f}(y) &= f(\Psi(y)) - a_{ij}(\Psi(y))\Phi^l_{x_i x_j}(\Psi(y))D_{y_l}\tilde{u}. \end{aligned}$$

We now recall the imposed conditions on $\mathbf{A}$ and $\partial\Omega$ and the a priori assumption (2.2.23), to observe that $\tilde{f} \in L^p_{\tilde{w}}(B_{6s}^+)$. We also check that the resulting matrix

$$\tilde{\mathbf{A}}(y) = (\tilde{a}_{lm}(y)) = [\bigtriangledown\Phi(\Psi(y))] \cdot \mathbf{A}(\Psi(y)) \cdot [\bigtriangledown\Phi(\Psi(y))]^t$$

satisfies a small BMO assumption. Indeed, note from the conditions on $\mathbf{A}$ and $\partial\Omega$ that

$$\begin{aligned} \|\tilde{\mathbf{A}}\|_* &\le c\left(\|\mathbf{A}\|_* + \|\bigtriangledown\gamma\|_{L^\infty\left(B_r'(x_0')\right)} + \|\bigtriangledown\gamma\|^2_{L^\infty\left(B_r'(x_0')\right)}\right) \\ &\le c\left(\delta + r\|\bigtriangledown^2\gamma\|_{L^\infty\left(B_r'(x_0')\right)} + r^2\|\bigtriangledown^2\gamma\|^2_{L^\infty\left(B_r'(x_0')\right)}\right) \\ &\le c(\delta + r + r^2), \end{aligned}$$

where $B_\rho'(x') := \{y' \in \mathbb{R}^{n-1} : |y' - x'| < \rho\}$ is a ball in $\mathbb{R}^{n-1}$. Then we choose $\delta = \delta(n, \Lambda, \gamma) > 0$ and $r = r(n, \Lambda, \gamma) > 0$ sufficiently small so that all the hypotheses of Theorem 2.2.16 with $\frac{\tilde{u}(sy)}{s^2}$, $\tilde{\mathbf{A}}(sy)$, $\tilde{f}(sy)$ and $\tilde{w}(sy)$ for $y \in B_6^+$ are satisfied. In turn, we apply Theorem 2.2.16 to discover that $|D^2\tilde{u}|^2 \in L_{\tilde{w}}^{\frac{p}{2}}(B_s^+)$ with the estimate

$$\fint_{B_s^+} |D^2\tilde{u}|^p \tilde{w} dy \le c \left(\underbrace{\fint_{B_{6s}^+} |\tilde{f}|^p \tilde{w} dy}_{I_1} + \underbrace{\frac{1}{s^{2p}} \left[\fint_{B_{6s}^+} |\tilde{u}|^2 dy \right]^{\frac{p}{2}}}_{I_2} \right). \tag{2.2.24}$$

We recall $\partial\Omega \in C^{1,1}$ and $\tilde{u} \in W_w^{2,p}(\Omega)$ in order to derive

$$I_1 \le \fint_{B_{6s}^+} |f(\Psi)|^p \tilde{w} dy + c \fint_{B_{6s}^+} |D\tilde{u}|^p \tilde{w} dy.$$

On the other hand, we recall $\tilde{w} \in A_{\frac{p}{2}}$ and use Hölder's inequality to find

$$\begin{aligned} I_2 &\le \frac{1}{s^{2p} |B_{6s}^+|^{\frac{p}{2}}} \left(\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w} dy \right) \left(\int_{B_{6s}^+} \tilde{w}^{\frac{-2}{p-2}} dy \right)^{\frac{p-2}{2}} \\ &\le \frac{[\tilde{w}]_{\frac{p}{2}}}{s^{2p} \tilde{w}(B_{6s}^+)} \left(\int_{B_{6s}^+} |\tilde{u}|^p \tilde{w} dy \right) \\ &\le \frac{[\tilde{w}]_{\frac{p}{2}}^2 |B_6|^{\frac{p}{2}}}{s^{2p} \tilde{w}(B_6) |B_{6s}^+|^{\frac{p}{2}}} \left(\int_{B_{6s}^+} |\tilde{u}|^p \, \tilde{w} dy \right) \\ &\le \frac{c}{s^{p(2+\frac{n}{2})}} \int_{B_{6s}^+} |\tilde{u}|^p \, \tilde{w} dy \end{aligned}$$

from a direct computation using Lemma 2.2.1 that

$$\frac{\tilde{w}(B_{6s}^+)}{\tilde{w}(B_6)} \ge [\tilde{w}]_{\frac{p}{2}}^{-1} \left(\frac{|B_{6s}^+|}{|B_6|} \right)^{\frac{p}{2}}.$$

Consequently, inserting the above resulting estimates of I_1 and I_2 into (2.2.24), we obtain

$$\int_{B_s^+} |D^2\tilde{u}|^p \tilde{w} dy \le c \left(\int_{B_{6s}^+} |f(\Psi)|^p \tilde{w} dy + \int_{B_{6s}^+} |D\tilde{u}|^p \tilde{w} dy + \frac{1}{s^{2p}} \int_{B_{6s}^+} |\tilde{u}|^p \, \tilde{w} dy \right).$$

Converting back to the x-variables, we conclude

$$\begin{aligned}
&\int_{V_s} |D^2u|^p w dx \\
&\leq c\left(\int_{\Psi(B_{6s}^+)} |f|^p w dx + \int_{\Psi(B_{6s}^+)} |Du|^p w dx + \frac{1}{s^{2p}}\int_{\Psi(B_{6s}^+)} |u|^p w dx\right) \\
&\leq c\left(\int_{\Omega} |f|^p w dx + \int_{\Omega} |Du|^p w dx + \int_{\Omega} |u|^p w dx\right),
\end{aligned}$$

where $V_s := \Psi(B_s^+)$. Since $\partial\Omega$ is compact, we can cover $\partial\Omega$ by a finite number of sets $V_{s_1}, V_{s_2}, \cdots, V_{s_N}$ as above and find a finite number of small positive constants $s_1, s_2, \cdots, s_N$. We therefore have, by summing the resulting estimates, along with the interior estimate over some open set $V_{s_0} \subset\subset \Omega$ so that $\Omega \subset \bigcup_{i=0}^N V_{s_i}$, that

$$|D^2u|^2 \in L_w^{\frac{p}{2}}(\Omega)$$

with the estimate

$$\int_\Omega |D^2u|^p w dx \leq c\left(\int_\Omega |f|^p w dx + \int_\Omega |Du|^p w dx + \int_\Omega |u|^p w dx\right).$$

Then we use the weighted Sobolev interpolation inequality in [26] to find

$$\int_\Omega |D^2u|^p w dx \leq c\left(\int_\Omega |f|^p w dx + \tau\int_\Omega |D^2u|^p w dx + (1+c(\tau))\int_\Omega |u|^p w dx\right)$$

for any small $\tau > 0$. By taking τ small enough so that $c\tau \leq \frac{1}{2}$, we arrive at

$$\int_\Omega |D^2u|^p w dx \leq c\left(\int_\Omega |f|^p w dx + \int_\Omega |u|^p w dx\right).$$

In addition, using the uniqueness of $W^{2,p}$ solutions, we eventually obtain the desired estimate

$$\int_\Omega |D^2u|^p w dx \leq c\int_\Omega |f|^p w dx. \tag{2.2.25}$$

Now it remains to remove the a priori assumption (2.2.23). To this end, select a sequence $\{a_{ij}^k\}_{k=1}^\infty$ of smooth functions with uniform (δ, R)-vanishing property such that

$$a_{ij}^k \to a_{ij} \quad \text{in } L^t(\Omega) \qquad \text{for each } 1 < t < \infty. \tag{2.2.26}$$

We also take a sequence $\{f^k\}_{k=1}^\infty$ of smooth functions in $C_0^\infty(\Omega)$ satisfying

$$f^k \to f \quad \text{in} \quad L_w^p(\Omega) \qquad \text{and} \qquad \|f^k\|_{L_w^p(\Omega)} \le \|f\|_{L_w^p(\Omega)} + 1. \tag{2.2.27}$$

Then there exists a unique solution $u^k \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ of

$$\begin{cases} a_{ij}^k D_{ij} u^k &= f^k \quad \text{in} \quad \Omega, \\ u^k &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.2.28}$$

for any $2 < q < \infty$. Needless to say, these solutions u^k are in $W_w^{2,p}(\Omega)$. But then from the estimate (2.2.25), we have

$$\|D^2 u^k\|_{L_w^p(\Omega)} \le c\|f^k\|_{L_w^p(\Omega)}, \tag{2.2.29}$$

where c is independent of k. Thus it follows from (2.2.27) and (2.2.29) that

$$\|D^2 u^k\|_{L_w^p(\Omega)} \le c\left(\|f\|_{L_w^p(\Omega)} + 1\right). \tag{2.2.30}$$

On the other hand, we recall the interpolation inequality in [26], and then use the weighted Poincaré inequality, to discover

$$\begin{aligned} \int_\Omega |Du^k|^p w dx &\le \tau \int_\Omega |u^k|^p w dx + c(\tau) \int_\Omega |D^2 u^k|^p w dx \\ &\le c\tau \int_\Omega |Du^k|^p w dx + c(\tau) \int_\Omega |D^2 u^k|^p w dx. \end{aligned}$$

We then select small $\tau > 0$ to derive

$$\|Du^k\|_{L_w^p(\Omega)} \le c\|D^2 u^k\|_{L_w^p(\Omega)} \le c\left(\|f\|_{L_w^p(\Omega)} + 1\right).$$

This estimate and the weighted Poincaré inequality imply

$$\|u^k\|_{L_w^p(\Omega)} \le c\|Du^k\|_{L_w^p(\Omega)} \le c\|D^2 u^k\|_{L_w^p(\Omega)} \le c\left(\|f\|_{L_w^p(\Omega)} + 1\right),$$

and thus $\{u^k\}_{k=1}^\infty$ is uniformly bounded in $W_w^{2,p}(\Omega)$. Then there exist a subsequence, which we still denote by $\{u^k\}_{k=1}^\infty$, and a function $v \in W_w^{2,p}(\Omega)$

such that

$$u^k \rightharpoonup v \ \text{ weakly in } W^{2,p}_w(\Omega). \tag{2.2.31}$$

In view of (2.2.26)-(2.2.28) and (2.2.31), we easily observe that v is also a solution of (2.0.1). Then by the uniqueness for the problem (2.0.1) we conclude $u = v$. Hence the proof is completed. □

2.3 $W^{2,p(\cdot)}$-estimates

2.3.1 Preliminaries and main result

We introduce the variable exponent Lebesgue and Sobolev spaces. Let us consider a measurable function $p(\cdot) = p(x) : \mathbb{R}^n \to \mathbb{R}$, which is called the *variable exponent*, satisfying

$$1 < \gamma_1 \le p(x) \le \gamma_2 < \infty, \ \forall x \in \mathbb{R}^n, \tag{2.3.1}$$

for some constants γ_1 and γ_2. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $g : \Omega \to \mathbb{R}$ such that the *modular*

$$\varrho_{p(\cdot)}(g) := \int_\Omega |g|^{p(x)} dx$$

is finite. From the assumption (2.3.1), this space $L^{p(\cdot)}(\Omega)$ is a reflexive Banach space equipped with the following *Luxemburg norm*

$$\|g\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{g}{\lambda}\right) \le 1\right\},$$

with its dual space $L^{p'(\cdot)}(\Omega)$, where $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. The *variable exponent Sobolev space* $W^{k,p(\cdot)}(\Omega)$ consists of functions $g \in L^{p(\cdot)}(\Omega)$ whose distributional derivatives $D^\alpha g$ also belong to $L^{p(\cdot)}(\Omega)$ for all α with $|\alpha| \le k$, and its norm of g in $W^{k,p(\cdot)}(\Omega)$ is given by

$$\|g\|_{W^{k,p(\cdot)}(\Omega)} = \sum_{0 \le |\alpha| \le k} \|D^\alpha g\|_{L^{p(\cdot)}(\Omega)}.$$

As in the case of classical Sobolev spaces, $W_0^{1,p(\cdot)}(\Omega)$ is taken to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For the sake of simplicity, we write $\|Dg\|_{L^{p(\cdot)}(\Omega)} = \||Dg|\|_{L^{p(\cdot)}(\Omega)}$ and $\|D^2 g\|_{L^{p(\cdot)}(\Omega)} = \||D^2 g|\|_{L^{p(\cdot)}(\Omega)}$.

Note that $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and

$$\|g\|_{L^{p(\cdot)}(\Omega)} \leq 1 \text{ if and only if } \varrho_{p(\cdot)}(g) \leq 1, \tag{2.3.2}$$

which is called the norm-modular unit ball property. Also, we have the following relation between norm and modular:

$$\min\left\{\varrho_{p(\cdot)}(g)^{\frac{1}{\gamma_1}}, \varrho_{p(\cdot)}(g)^{\frac{1}{\gamma_2}}\right\} \leq \|g\|_{L^{p(\cdot)}(\Omega)} \leq \max\left\{\varrho_{p(\cdot)}(g)^{\frac{1}{\gamma_1}}, \varrho_{p(\cdot)}(g)^{\frac{1}{\gamma_2}}\right\}. \tag{2.3.3}$$

We now present an important assumption on $p(\cdot)$. Suppose that $p(\cdot)$ is uniformly continuous with modulus of continuity ω, that is,

$$|p(x) - p(y)| \leq \omega(|x-y|), \quad \forall x, y \in \mathbb{R}^n, \tag{2.3.4}$$

where $\omega : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function with $\omega(0) = 0$. Moreover, the modulus continuity ω satisfies that

$$\omega(\rho)\log\left(\frac{1}{\rho}\right) \leq M, \quad \forall \rho \in (0,1), \tag{2.3.5}$$

for some constant $M = M(\omega(\cdot)) > 0$. We point out that the conditions (2.3.4) and (2.3.5) are equivalent to *log-Hölder continuity* of $p(\cdot)$ in the bounded domain Ω, that is, there exists $M_1 > 0$ such that

$$|p(x) - p(y)| \leq \frac{M_1}{-\log|x-y|}, \quad \text{for all } x, y \in \Omega \text{ with } |x-y| \leq \frac{1}{2}. \tag{2.3.6}$$

We refer to [28] for further discussion on variable exponent spaces.

We now state one of the main theorems in this chapter.

Theorem 2.3.1 (Main Theorem)**.** *Suppose that* $p(\cdot)$ *satisfies* (2.3.1), (2.3.4) *and* (2.3.5). *Assume* $\partial\Omega \in C^{1,1}$ *and* $f \in L^{p(\cdot)}(\Omega)$. *Then there exists a positive small* $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \partial\Omega)$ *so that if* $\mathbf{A}$ *is* (δ, R)*-vanishing for some* $R > 0$, *then the problem (2.0.1) has a unique strong solution* $u \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ *with the estimate*

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \leq c\|f\|_{L^{p(\cdot)}(\Omega)}, \tag{2.3.7}$$

for some positive constant $c = c(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \partial\Omega, diam(\Omega), R)$.

A strong solution of the equation (2.0.1), which is treated throughout the thesis, is a twice weakly differentiable function satisfying the equation

(2.0.1) almost everywhere in Ω and assuming boundary values on $\partial\Omega$ in the trace sense.

Using the linearity of the equation (2.0.1), we can directly obtain the following result from the above theorem.

Corollary 2.3.2. *Suppose that $p(\cdot)$ satisfies (2.3.1), (2.3.4) and (2.3.5). Assume $\partial\Omega \in C^{1,1}$, $f \in L^{p(\cdot)}(\Omega)$ and $\phi \in W^{2,p(\cdot)}(\Omega)$. Then there exists a positive small $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \partial\Omega)$ so that if $\mathbf{A}$ is (δ, R)-vanishing for some $R > 0$, then the problem*

$$\begin{cases} a_{ij} D_{ij} u &= f \quad \text{in} \quad \Omega, \\ \qquad u &= \phi \quad \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution $u \in W^{2,p(\cdot)}(\Omega)$ with $u - \phi \in W_0^{1,p(\cdot)}(\Omega)$, and we have the estimate

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \leq c \left(\|f\|_{L^{p(\cdot)}(\Omega)} + \|\phi\|_{W^{2,p(\cdot)}(\Omega)} \right), \tag{2.3.8}$$

for some positive constant $c = c(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \partial\Omega, diam(\Omega), R)$.

We end this chapter with the following well known $W^{2,q}$ regularity results with $1 < q < \infty$, which will be used later, see [24, 25] for details.

Lemma 2.3.3. *Let $1 < q < \infty$. There exist $\delta = \delta(\Lambda, n, q) > 0$ and $c = c(\Lambda, n, q) > 0$ such that for any fixed $r > 0$,*

(i) (Interior estimates) if $\mathbf{A}$ is $(\delta, 2r)$-vanishing and $f \in L^q(B_{2r})$, then for any strong solution $u \in W^{2,q}(B_{2r})$ of

$$a_{ij} D_{ij} u = f \quad \text{in} \ \ B_{2r},$$

we have the estimate

$$\int_{B_r} |D^2 u|^q dx \leq c \left(\int_{B_{2r}} |f|^q dx + \frac{1}{r^{2q}} \int_{B_{2r}} |u|^q dx \right),$$

(ii) (Boundary estimates) if $\mathbf{A}$ is $(\delta, 2r)$-vanishing and $f \in L^q(B_{2r}^+)$, then for any strong solution $u \in W^{2,q}(B_{2r}^+)$ of

$$\begin{cases} a_{ij} D_{ij} u &= f \quad \text{in} \quad B_{2r}^+, \\ \qquad u &= 0 \quad \text{on} \quad T_{2r}, \end{cases}$$

we have the estimate

$$\int_{B_r^+} |D^2u|^q dx \le c \left(\int_{B_{2r}^+} |f|^q dx + \frac{1}{r^{2q}} \int_{B_{2r}^+} |u|^q dx \right).$$

2.3.2 Interior and boundary $W^{2,p(\cdot)}$-estimates

To prove the global $W^{2,p(\cdot)}$-estimate, we shall use standard covering and flattening arguments after deriving interior and boundary estimates. Therefore, in this section, we establish the a priori $W^{2,p(\cdot)}$-estimates on balls and half balls, see Theorem 2.3.4.

Now, we state the main results in this section.

Theorem 2.3.4. *Suppose that* $p(\cdot)$ *satisfies* (2.3.1), (2.3.4) *and* (2.3.5). *Let* $\rho_0 > 0$ *be the largest number fulfilled*

$$\rho_0 \le \frac{1}{2}, \quad |B_{4\rho_0}| \le 1 \quad \textit{and} \quad \omega(4\rho_0) \le \min\left\{ \frac{\gamma_1 - 1}{2}, 1 \right\}. \tag{2.3.9}$$

There exist $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 0$ *and* $c = c(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 1$ *such that for any fixed* $\rho \in (0, \rho_0]$,

(i) (Interior estimates) if $\mathbf{A}$ *is* $(\delta, 4\rho)$*-vanishing and if* $f \in L^{p(\cdot)}(B_{4\rho})$, *then for any solution* $u \in W^{2,p(\cdot)}(B_{4\rho})$ *of*

$$a_{ij} D_{ij} u = f \quad \textit{in } B_{4\rho},$$

we have the estimate

$$\|D^2u\|_{L^{p(\cdot)}(B_\rho)} \le c \rho^{-\frac{n(\gamma_2 - \gamma_1 + \omega(4\rho_0))}{\gamma_1(\gamma_1 - \omega(4\rho_0))}} \left(\|f\|_{L^{p(\cdot)}(B_{4\rho})} + \frac{1}{\rho^2} \|u\|_{L^{\gamma_1}(B_{4\rho})} \right), \tag{2.3.10}$$

(ii) (Boundary estimates) if $\mathbf{A}$ *is* $(\delta, 4\rho)$*-vanishing and if* $f \in L^{p(\cdot)}(B_{4\rho}^+)$, *then for any solution* $u \in W^{2,p(\cdot)}(B_{4\rho}^+)$ *of*

$$\begin{cases} a_{ij} D_{ij} u &= f \quad \textit{in} \quad B_{4\rho}^+, \\ u &= 0 \quad \textit{on} \quad T_{4\rho}, \end{cases} \tag{2.3.11}$$

we have the estimate

$$\|D^2u\|_{L^{p(\cdot)}(B_\rho^+)} \le c \rho^{-\frac{n(\gamma_2 - \gamma_1 + \omega(4\rho_0))}{\gamma_1(\gamma_1 - \omega(4\rho_0))}} \left(\|f\|_{L^{p(\cdot)}(B_{4\rho}^+)} + \frac{1}{\rho^2} \|u\|_{L^{\gamma_1}(B_{4\rho}^+)} \right). \tag{2.3.12}$$

Here, we only establish the boundary estimates in Theorem 2.3.4, since the interior estimates can be derived in the same way to the boundary estimates.

In order to obtain the boundary estimates , our aim is to show

$$\|D^2u\|_{L^{p(\cdot)}(B^+_\rho)} \le c\rho^{-\frac{n(\gamma_2-\gamma_1+\omega(4\rho_0))}{\gamma_1(\gamma_1-\omega(4\rho_0))}}, \tag{2.3.13}$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 1$, under the assumptions that

$$\|f\|_{L^{p(\cdot)}(B^+_{4\rho})} \le 1 \quad \text{and} \quad \|u\|_{L^{\gamma_1}(B^+_{4\rho})} \le \rho^2. \tag{2.3.14}$$

Indeed, for u and f given in the assumption of Theorem 2.3.4 (ii), we consider

$$\tilde{u} = \frac{u}{\left(\|f\|_{L^{p(\cdot)}(B^+_{4\rho})} + \frac{1}{\rho^2}\|u\|_{L^{\gamma_1}(B^+_{4\rho})}\right)} \quad \text{and} \quad \tilde{f} = \frac{f}{\left(\|f\|_{L^{p(\cdot)}(B^+_{4\rho})} + \frac{1}{\rho^2}\|u\|_{L^{\gamma_1}(B^+_{4\rho})}\right)}.$$

Then it is clear that $\tilde{u}$ is a solution of

$$\begin{cases} a_{ij}D_{ij}\tilde{u} &= \tilde{f} \quad \text{in} \quad B^+_{4\rho}, \\ \tilde{u} &= 0 \quad \text{on} \quad T_{4\rho}, \end{cases}$$

with $\|\tilde{f}\|_{L^{p(\cdot)}(B^+_{4\rho})} \le 1$ and $\|\tilde{u}\|_{L^{\gamma_1}(B^+_{4\rho})} \le \rho^2$. Therefore, in view of (2.3.13) and (2.3.14), we have

$$\|D^2\tilde{u}\|_{L^{p(\cdot)}(B^+_\rho)} \le c\rho^{-\frac{n(\gamma_2-\gamma_1+\omega(4\rho_0))}{\gamma_1(\gamma_1-\omega(4\rho_0))}},$$

which implies the desired estimate (2.3.12).

Before we get to the proof of Theorem 2.3.4 (ii), we need to prove a series of lemmas. Hereafter, in this section, we fix $\rho \le \rho_0$ and assume that $\mathbf{A}$ is $(\delta, 4\rho)$-vanishing and $u \in W^{1,p(\cdot)}(B^+_{4\rho})$ is a solution of (2.3.11) with (2.3.14). We also denote by c to mean any positive constant being dependent only on $n, \Lambda, \gamma_1, \gamma_2$ and $\omega(\cdot)$.

We remark that applying Lemma 2.3.3 with $r = 2\rho$ and $q = \gamma_1$, we infer from (2.3.2), (2.3.9) and (2.3.14) that

$$\begin{aligned} \int_{B^+_{2\rho}} |D^2u|^{\gamma_1} dx &\le c\left(\int_{B^+_{4\rho}} |f|^{\gamma_1} dx + \frac{1}{(2\rho)^{2\gamma_1}} \int_{B^+_{4\rho}} |u|^{\gamma_1} dx\right) \\ &\le c\left(\int_{B^+_{4\rho}} \left[|f|^{p(x)} + 1\right] dx + \frac{1}{2^{2\gamma_1}}\right) \end{aligned}$$

$$\leq \quad c\left(1+|B_{4\rho}^{+}|+\frac{1}{2^{2\gamma_1}}\right)\leq c, \tag{2.3.15}$$

by taking sufficiently small $\delta=\delta(\Lambda,n,\gamma_1)>0$.

We first introduce some notations. Fix any s_1 and s_2 with $1\leq s_1<s_2\leq 2$. Set

$$p^{-}:=\inf_{x\in B_{2\rho}^{+}} p(x),\quad p^{+}:=\sup_{x\in B_{2\rho}^{+}} p(x)\ \text{ and }\ \gamma_0:=\gamma_1-\omega(4\rho_0). \tag{2.3.16}$$

Then it follows from (2.3.9) that

$$1<\frac{\gamma_1+1}{2}\leq\gamma_0\leq\gamma_1\leq p^{-}\leq p^{+}\leq\gamma_2<+\infty.$$

We write

$$\lambda_0:=\fint_{B_{2\rho}^{+}}\left[|D^2u|^{\frac{\gamma_0 p(x)}{p^{-}}}+\frac{1}{\delta}\left(|f|^{\frac{\gamma_0 p(x)}{p^{-}}}+1\right)\right]dx>1, \tag{2.3.17}$$

where $\delta\in(0,1)$ is a small constant which will be determined later. Since $\frac{\gamma_0 p(x)}{p^{-}}\leq\gamma_1(1-\frac{\omega(4\rho_0)}{\gamma_1})(1+\frac{\omega(4\rho)}{\gamma_1})\leq\gamma_1\leq p(x)$, it is clear that the above integral (2.3.17) is well defined. We then define an upper level set

$$E(\lambda):=\left\{x\in B_{s_1\rho}^{+}:|D^2u(x)|^{\frac{\gamma_0 p(x)}{p^{-}}}>\lambda\right\} \tag{2.3.18}$$

for λ large enough such that

$$\lambda\geq A\lambda_0,\ \text{ where } A:=\left(\frac{240}{s_2-s_1}\right)^{n}. \tag{2.3.19}$$

Lemma 2.3.5. *Given $\lambda\geq A\lambda_0$, there exists a disjoint family of $\left\{B_{\tau_k}^{+}(y^k)\right\}_{k=1}^{\infty}$ with $y^k\in E(\lambda)$ and $\tau_k\in\left(0,\frac{(s_2-s_1)\rho}{120}\right)$ such that*

$$E(\lambda)\subset\bigcup_{k=1}^{\infty}B_{5\tau_k}^{+}(y^k)$$

and

$$\Phi_{y^k}(\tau_k)=\lambda\quad\textit{and}\quad\Phi_{y^k}(\tau)<\lambda,\quad\textit{for all }\tau\in(\tau_k,(s_2-s_1)\rho] \tag{2.3.20}$$

where $\Phi_y : (0,(s_2-s_1)\rho] \to [0,\infty)$, $y \in E(\lambda)$, *is a continuous function defined by*

$$\Phi_y(\tau) = \fint_{B_\tau^+(y)} \left(|D^2u|^{\frac{\gamma_0 p(x)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(x)}{p^-}} \right) dx.$$

Proof. We first note that $B_\tau^+(y) \subset B_{s_2\rho}^+ \subset B_{2\rho}^+$ for all $0 < \tau \le (s_2-s_1)\rho$. Then we deduce from (2.3.19) that for any $\tau \in \left[\frac{(s_2-s_1)\rho}{120}, (s_2-s_1)\rho\right]$,

$$\begin{aligned}
\Phi_y(\tau) &= \fint_{B_\tau^+(y)} \left(|D^2u|^{\frac{\gamma_0 p(x)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(x)}{p^-}} \right) dx \\
&\le \frac{\left|B_{2\rho}^+\right|}{\left|B_\tau^+(y)\right|} \fint_{B_{2\rho}^+} \left(|D^2u|^{\frac{\gamma_0 p(x)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(x)}{p^-}} \right) dx \\
&= \left(\frac{2\rho}{\tau}\right)^n \fint_{B_{2\rho}^+} \left(|D^2u|^{\frac{\gamma_0 p(x)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(x)}{p^-}} \right) dx \\
&< \left(\frac{240}{s_2-s_1}\right)^n \fint_{B_{2\rho}^+} \left(|D^2u|^{\frac{\gamma_0 p(x)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(x)}{p^-}} \right) dx \\
&< A\lambda_0 \le \lambda.
\end{aligned}$$

However, for almost every $y \in E(\lambda)$ we have

$$\lim_{\tau\to 0} \Phi_y(\tau) = \lim_{\tau\to 0} \fint_{B_\tau^+(y)} \left(|D^2u|^{\frac{\gamma_0 p(x)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(x)}{p^-}} \right) dx > \lambda,$$

by Lebesgue's differentiation theorem.

Therefore, for almost every $y \in E(\lambda)$, we can find

$$\tau_y = \tau(y) \in \left(0, \frac{(s_2-s_1)\rho}{120}\right)$$

such that

$$\Phi_y(\tau_y) = \lambda \quad \text{and} \quad \Phi_y(\tau) < \lambda, \quad \text{for all } \tau \in (\tau_y, (s_2-s_1)\rho].$$

Accordingly, by the Vitali covering lemma, we can find the desired y^k and τ_k. $\square$

Furthermore, the following lemma immediately comes from Lemma 2.3.5.

Lemma 2.3.6. *Under the hypotheses and conclusions of Lemma 2.3.5, we have*

$$\left|B^+_{\tau_k}(y^k)\right| \le \frac{2}{\lambda}\left(\int_{B^+_{\tau_k}(y^k)\cap\left\{|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda}{4}\right\}}|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}\,dx\right.$$
$$\left.+\frac{1}{\delta}\int_{B^+_{\tau_k}(y^k)\cap\left\{|f|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}}|f|^{\frac{\gamma_0 p(x)}{p^-}}\,dx\right). \tag{2.3.21}$$

Proof. From Lemma 2.3.5, a direct calculation provides

$$\begin{aligned}
\left|B^+_{\tau_k}(y^k)\right| &= \frac{1}{\lambda}\int_{B^+_{\tau_k}(y^k)}\left(|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}+\frac{1}{\delta}|f|^{\frac{\gamma_0 p(x)}{p^-}}\right)dx\\
&\le \frac{1}{\lambda}\left(\int_{B^+_{\tau_k}(y^k)\cap\left\{|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda}{4}\right\}}|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}\,dx+\frac{\lambda}{4}\left|B^+_{\tau_k}(y^k)\right|\right.\\
&\qquad\left.+\frac{1}{\delta}\int_{B^+_{\tau_k}(y^k)\cap\left\{|f|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}}|f|^{\frac{\gamma_0 p(x)}{p^-}}\,dx+\frac{\lambda}{4}\left|B^+_{\tau_k}(y^k)\right|\right)\\
&= \frac{1}{2}\left|B^+_{\tau_k}(y^k)\right|+\frac{1}{\lambda}\left(\int_{B^+_{\tau_k}(y^k)\cap\left\{|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda}{4}\right\}}|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}\,dx\right.\\
&\qquad\left.+\frac{1}{\delta}\int_{B^+_{\tau_k}(y^k)\cap\left\{|f|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}}|f|^{\frac{\gamma_0 p(x)}{p^-}}\,dx\right),
\end{aligned}$$

and hence, we obtain the desired estimates (2.3.21). □

Now, we fix the point y^k and the scale τ_k on the results of Lemma 2.3.5. Then there are two possible cases, which we have to consider in our argument;

(i) (Interior case) $B_{20\tau_k}(y^k)\subset B^+_{s_2\rho}$,

(ii) (Boundary case) $B_{20\tau_k}(y^k)\not\subset B^+_{s_2\rho}$, which means $B_{20\tau_k}(y^k)\cap T_{s_2\rho}\neq\emptyset$.

Let us first investigate the interior case that $B_{20\tau_k}(y^k)\subset B^+_{s_2\rho}$. We set

$$p_k^- := \inf_{z\in B_{20\tau_k}(y^k)} p(z) \quad\text{and}\quad p_k^+ := \sup_{z\in B_{20\tau_k}(y^k)} p(z). \tag{2.3.22}$$

Recalling (2.3.4), we have

$$p_k^+ - p_k^- \leq \omega(40\tau_k). \tag{2.3.23}$$

Note that

$$40\tau_k \leq 2(s_2 - s_1)\rho \leq 2\rho_0 \quad \text{and} \quad B_{20\tau_k}(y^k) \subset B_{s_2}^+.$$

The following lemma will play an essential role in the proof of our results in this section.

Lemma 2.3.7. *(Interior case) Under the hypotheses as mentioned earlier, we have*

$$\fint_{B_{20\tau_k}(y^k)} |D^2 u|^{\gamma_0} dx \leq c_0 \lambda^{\frac{p^-}{p_k^+}} \quad \text{and} \quad \fint_{B_{20\tau_k}(y^k)} |f|^{\gamma_0} dx \leq c_0 \lambda^{\frac{p^-}{p_k^+}} \delta^{\frac{\gamma_1}{\gamma_2}}, \tag{2.3.24}$$

for some $c_0 = c_0(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 0$.

Moreover, for any $\epsilon \in (0,1)$*, there exist* $\delta = \delta(\epsilon, n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 0$ *and* $v_k \in W^{2,\gamma_0}(B_{15\tau_k}(y^k)) \cap W^{2,\infty}(B_{5\tau_k}(y^k))$ *such that*

$$\fint_{B_{5\tau_k}(y^k)} |D^2(u - v_k)|^{\gamma_0} dx \leq \epsilon c_0 \lambda^{\frac{p^-}{p_k^+}} \quad \text{and} \quad \|D^2 v_k\|^{\gamma_0}_{L^\infty\left(B_{5\tau_k}(y^k)\right)} \leq c_1 \lambda^{\frac{p^-}{p_k^+}}, \tag{2.3.25}$$

for some $c_1 = c_1(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 1$.

Proof. From (2.3.9), we know $40\tau_k \leq 1$, $\omega(40\tau_k) \leq 1$ and $|B_{20\tau_k}| \leq 1$. In view of (2.3.5), (2.3.15) and (2.3.23), we infer

$$\begin{aligned}
&\left(\fint_{B_{20\tau_k}(y^k)} |D^2 u|^{\gamma_0} dx\right)^{p_k^+ - p_k^-} \\
&\leq \left(\frac{1}{|B_{20\tau_k}(y^k)|} \int_{B_{2\rho}^+} \left[|D^2 u|^{\gamma_1} + 1\right] dx\right)^{p_k^+ - p_k^-} \\
&\leq c \left(\frac{1}{|B_{20\tau_k}(y^k)|}\right)^{\omega(40\tau_k)} \leq c \left(\frac{1}{40\tau_k}\right)^{n\omega(40\tau_k)} \leq c.
\end{aligned} \tag{2.3.26}$$

Using Jensen's inequality and the facts that $\gamma_1 \leq p_k^+$ and $p^- \leq p_k^-$, we deduce from (2.3.20) and (2.3.26) that

$$\begin{aligned}
&\fint_{B_{20\tau_k}(y^k)} |D^2u|^{\gamma_0} dx \\
&\quad = \left(\fint_{B_{20\tau_k}(y^k)} |D^2u|^{\gamma_0} dx\right)^{\frac{p_k^+ - p_k^-}{p_k^+}} \left(\fint_{B_{20\tau_k}(y^k)} |D^2u|^{\gamma_0} dx\right)^{\frac{p_k^-}{p_k^+}} \\
&\quad \le c \left(\fint_{B_{20\tau_k}(y^k)} |D^2u|^{\gamma_0} dx\right)^{\frac{p_k^-}{p_k^+}} \le c \left(\fint_{B_{20\tau_k}(y^k)} |D^2u|^{\frac{\gamma_0 p_k^-}{p^-}} dx\right)^{\frac{p^-}{p_k^+}} \\
&\quad \le c \left(\fint_{B_{20\tau_k}(y^k)} |D^2u|^{\frac{\gamma_0 p(x)}{p^-}} dx + 1\right)^{\frac{p^-}{p_k^+}} \le c\lambda^{\frac{p^-}{p_k^+}}.
\end{aligned}$$

Similarly, since

$$\left(\fint_{B_{20\tau_k}(y^k)} |f|^{\gamma_0} dx\right)^{p_k^+ - p_k^-} \le c \left(\frac{1}{|B_{20\tau_k}(y^k)|}\right)^{\omega(40\tau_k)} \le c,$$

we derive

$$\begin{aligned}
\fint_{B_{20\tau_k}(y^k)} |f|^{\gamma_0} dx \ &\le\ c \left(\fint_{B_{20\tau_k}(y^k)} |f|^{\frac{\gamma_0 p(x)}{p^-}} dx + 1\right)^{\frac{p^-}{p_k^+}} \\
&\le\ c(\delta\lambda + 1)^{\frac{p^-}{p_k^+}} \le c(\delta\lambda + \delta\lambda_0)^{\frac{p^-}{p_k^+}} \le c\lambda^{\frac{p^-}{p_k^+}} \delta^{\frac{\gamma_1}{\gamma_2}},
\end{aligned}$$

where the third inequality follows from (2.3.17). Therefore we finally get the desired estimates (2.3.24).

Now, let

$$w(y) := \frac{u(5\tau_k(y - y^k))}{(5\tau_k)^2 \left(c_0 \lambda^{\frac{p^-}{p_k^+}}\right)^{\frac{1}{\gamma_0}}}, \quad g(y) := \frac{f(5\tau_k(y - y^k))}{\left(c_0 \lambda^{\frac{p^-}{p_k^+}}\right)^{\frac{1}{\gamma_0}}},$$

and

$$(b_{ij}(y)) = \mathbf{B}(y) := \mathbf{A}(5\tau_k(y - y^k)).$$

Then it is easy to see that $w \in W^{2,p(\cdot)}(B_4) \subset W^{2,\gamma_0}(B_4)$ is a solution of

$$b_{ij} D_{ij} w = g \quad \text{in } B_4, \tag{2.3.27}$$

and moreover, we have from the $(\delta, 4\rho)$-vanishing condition of $\mathbf{A}$ and (2.3.24) that

$$[\mathbf{B}]_4 \leq \delta, \quad \fint_{B_4} |D^2 w|^{\gamma_0} dy \leq 1 \quad \text{and} \quad \fint_{B_4} |g|^{\gamma_0} dy \leq \delta^{\frac{\gamma_1}{\gamma_2}}.$$

We apply Lemma 2.1.5 and Corollary 2.1.6 to the above equation (2.3.27) with q and δ replaced by γ_0 and $\delta^{\frac{\gamma_1}{\gamma_2}}$, respectively, in order to discover that there exists a solution $v \in W^{2,\gamma_0}(B_3)$ of

$$\overline{b_{ij}}_{B_4} D_{ij} v = 0 \quad \text{in } B_3,$$

with

$$\fint_{B_1} |D^2(w-v)|^{\gamma_0} dy \leq \epsilon \quad \text{and} \quad \|D^2 v\|^{\gamma_0}_{L^\infty(B_1)} \leq c_1,$$

for some $c_1 = c_1(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 1$. Thus,

$$v_k(x) := (5\tau_k)^2 \left(c_0 \lambda^{\frac{p^-}{p_k^+}} \right)^{\frac{1}{\gamma_0}} v\left(y^k + \frac{1}{5\tau_k} x \right)$$

belongs to

$$W^{2,\gamma_0}(B_{15\tau_k}(y^k)) \cap W^{2,\infty}(B_{5\tau_k}(y^k))$$

and satisfies (2.3.25). $\square$

Next we examine the boundary case that $B_{20\tau_k}(y^k) \cap T_{s_2\rho} \neq \emptyset$. In this case, we write

$$\tilde{y}^k := (y^{k\prime}, 0), \quad \text{where } y^k = (y_1^k, \ldots, y_n^k) = (y^{k\prime}, y_n^k).$$

Then, we see that $|y^k - \tilde{y}^k| < 20\tau_k$. Since $120\tau_k \leq (s_2 - s_1)\rho \leq \rho_0$, we have

$$B_{5\tau_k}(y^k) \subset B^+_{25\tau_k}(\tilde{y}^k) \subset B^+_{100\tau_k}(\tilde{y}^k) \subset B^+_{120\tau_k}(y^k) \subset B^+_{s_2\rho}. \tag{2.3.28}$$

Set

$$p_k^- := \inf_{z \in B^+_{100\tau_k}(\tilde{y}^k)} p(z) \quad \text{and} \quad p_k^+ := \sup_{z \in B^+_{100\tau_k}(\tilde{y}^k)} p(z), \tag{2.3.29}$$

then we know from (2.3.4) that

$$p_k^+ - p_k^- \le \omega(200\tau_k).$$

Lemma 2.3.8. *Under the hypotheses as mentioned earlier, we have*

$$\fint_{B^+_{100\tau_k}(\tilde{y}^k)} |D^2u|^{\gamma_0} dx \le c_2 \lambda^{\frac{p^-}{p_k^+}} \quad \text{and} \quad \fint_{B^+_{100\tau_k}(\tilde{y}^k)} |f|^{\gamma_0} dx \le c_2 \lambda^{\frac{p^-}{p_k^+}} \delta^{\frac{\gamma_1}{\gamma_2}},$$

for some $c_2 = c_2(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 0$.

Moreover, for any $\epsilon \in (0,1)$, there exist $\delta = \delta(\epsilon, n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 0$ and $v_k \in W^{2,\gamma_0}(B^+_{75\tau_k}(\tilde{y}^k)) \cap W^{2,\infty}(B^+_{25\tau_k}(\tilde{y}^k))$ such that

$$\fint_{B^+_{25\tau_k}(\tilde{y}^k)} |D^2(u - v_k)|^{\gamma_0} dx \le \epsilon c_2 \lambda^{\frac{p^-}{p_k^+}} \text{ and } \|D^2 v_k\|^{\gamma_0}_{L^\infty\left(B^+_{25\tau_k}(y^k)\right)} \le c_3 \lambda^{\frac{p^-}{p_k^+}}, \tag{2.3.30}$$

for some $c_3 = c_3(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 1$.

Proof. From (2.3.20) we know

$$\fint_{B^+_{120\tau_k}(y^k)} |D^2u|^{\frac{\gamma_0 p(x)}{p^-}} dx \le \lambda \text{ and } \fint_{B^+_{120\tau_k}(y^k)} |f|^{\frac{\gamma_0 p(x)}{p^-}} dx \le \delta\lambda,$$

which, together with (2.3.28), implies that

$$\fint_{B^+_{100\tau_k}(\tilde{y}^k)} |D^2u|^{\frac{\gamma_0 p(x)}{p^-}} dx \le 2^n\lambda \text{ and } \fint_{B^+_{100\tau_k}(\tilde{y}^k)} |f|^{\frac{\gamma_0 p(x)}{p^-}} dx \le 2^n\delta\lambda. \tag{2.3.31}$$

The rest of the proof is similar to that of Lemma 2.3.7, by using (2.3.31), Lemma 2.1.7 and Corollary 2.1.8, in stead of (2.3.20), Lemma 2.1.5 and Corollary 2.1.6, respectively, so we omit here. $\square$

We recall the following basic identities and inequality that will be used in the proof of Theorem 2.3.4.

For $g \in L^t(\Omega)$ with $t > 1$,

$$\int_\Omega |g(x)|^t \, dx = \int_0^\infty t\lambda^{t-1} \left|\{x \in \Omega : |g(x)| > \lambda\}\right| d\lambda. \tag{2.3.32}$$

Applying the *Fubini theorem* to the above identity, we discover

$$\int_\Omega |g(x)|^t \, dx = \int_0^\infty (t-\tilde{t})\lambda^{t-\tilde{t}-1} \int_{\{x\in\Omega:|g(x)|>\lambda\}} |g(x)|^{\tilde{t}} \, dx d\lambda, \tag{2.3.33}$$

for $t > \tilde{t} \geq 1$. We also note the inequality

$$|\{x \in \Omega : |g(x)| > \lambda\}| \leq \frac{1}{\lambda}\int_\Omega |g(x)|dx, \tag{2.3.34}$$

for a function $g \in L^1(\Omega)$.

The following lemma will be used later in the exit time argument which is presented in the proof of Theorem 2.3.4 and has been introduced in [59].

Lemma 2.3.9. *(see [41]) Let $h : [a,b] \to \mathbb{R}$ be a bounded nonnegative function and suppose that for any s_1, s_2 with $0 < a \leq s_1 < s_2 \leq b$,*

$$h(s_1) \leq \theta h(s_2) + \frac{\sigma_1}{(s_2-s_1)^\beta} + \sigma_2,$$

where $\sigma_1, \sigma_2 \geq 0, \beta > 0$ and $0 \leq \theta < 1$. Then we have

$$h(s_1) \leq c\left(\frac{\sigma_1}{(s_2-s_1)^\beta} + \sigma_2\right),$$

for some constant $c = c(\beta, \theta) > 0$.

Now we are ready to prove Theorem 2.3.4.

Proof of Theorem 2.3.4 (ii). We set

$$K := (2^{\gamma_0-1}c_4)^{\frac{\gamma_2}{\gamma_1}} \quad \text{and} \quad c_4 := \max\{c_1, c_3\}, \tag{2.3.35}$$

where c_1 and c_3 are given in Lemma 2.3.7 and Lemma 2.3.8, respectively.

We first estimate the measure of the upper level set $E(K\lambda)$ given by (2.3.18), for all $\lambda \geq A\lambda_0$. Since $K \geq 1$, we see $E(K\lambda) \subset E(\lambda)$. Recalling Lemma 2.3.5, there exists a disjoint family of $\left\{B^+_{\tau_k}(y^k)\right\}_{k=1}^\infty$ with $y^k \in E(\lambda)$ and $\tau_k \in \left(0, \frac{(s_2-s_1)\rho}{120}\right)$ such that

$$E(K\lambda) \subset E(\lambda) \subset \bigcup_{k=1}^\infty B^+_{5\tau_k}(y^k),$$

and then we have

$$\begin{aligned} |E(K\lambda)| &= \left|\left\{x \in B^+_{s_1\rho} : \left|D^2u(x)\right|^{\frac{\gamma_0 p(x)}{p^-}} > K\lambda\right\}\right| \\ &\le \sum_{k=1}^{\infty}\left|\left\{x \in B^+_{5\tau_k}(y^k) : \left|D^2u(x)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(x)}}\right\}\right|. \end{aligned} \tag{2.3.36}$$

Let us first consider the interior case that $B^+_{5\tau_k}(y^k) \subset B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$, which implies $B^+_{5\tau_k}(y^k) = B_{5\tau_k}(y^k)$. Using the elementary inequality $(a+b)^t \le 2^{t-1}(a^t+b^t)$ for any $a,b>0$ and $t \ge 1$ and (2.3.34), we deduce from (2.3.22), (2.3.25) and (2.3.35) that

$$\begin{aligned} &\left|\left\{x \in B^+_{5\tau_k}(y^k) : \left|D^2u(x)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(x)}}\right\}\right| \\ &\qquad \le \left|\left\{x \in B_{5\tau_k}(y^k) : \left|D^2(u-v_k)(x)\right|^{\gamma_0} > c_1\lambda^{\frac{p^-}{p_k^+}}\right\}\right| \\ &\qquad\quad + \left|\left\{x \in B_{5\tau_k}(y^k) : \left|D^2v_k(x)\right|^{\gamma_0} > c_1\lambda^{\frac{p^-}{p_k^+}}\right\}\right| \\ &\qquad \le \left(c_1\lambda^{\frac{p^-}{p_k^+}}\right)^{-1}\int_{B_{5\tau_k}(y^k)}\left|D^2(u-v_k)(x)\right|^{\gamma_0}dx \\ &\qquad \le \frac{\epsilon c_0}{c_1}\left|B_{5\tau_k}(y^k)\right| \le \epsilon c\left|B^+_{\tau_k}(y^k)\right|. \end{aligned} \tag{2.3.37}$$

Next we consider the boundary case that $B_{20\tau_k}(y^k) \cap T_{s_2\rho} \ne \emptyset$. In a similar way to (2.3.37), recall (2.3.28)-(2.3.30) and (2.3.35) to find

$$\begin{aligned} &\left|\left\{x \in B^+_{5\tau_k}(y^k) : \left|D^2u(x)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(x)}}\right\}\right| \\ &\qquad \le \left(c_3\lambda^{\frac{p^-}{p_k^+}}\right)^{-1}\int_{B^+_{25\tau_k}(\tilde{y}^k)}\left|D^2(u-v_k)(x)\right|^{\gamma_0}dx \\ &\qquad \le \frac{\epsilon c_2}{c_3}\left|B^+_{25\tau_k}(\tilde{y}^k)\right| \le \epsilon c\left|B^+_{\tau_k}(y^k)\right|. \end{aligned} \tag{2.3.38}$$

Consequently, combining (2.3.37) and (2.3.38) with (2.3.36), it follows from

(2.3.21) in Lemma 2.3.6 that

$$\begin{aligned}|E(K\lambda)| &\leq \epsilon c \sum_{k=1}^{\infty} \left| B^+_{\tau_k}(y^k) \right| \\ &\leq \frac{\epsilon c}{\lambda} \sum_{k=1}^{\infty} \left(\int_{B^+_{\tau_k}(y^k) \cap \left\{ |D^2 u|^{\frac{\gamma_0 p(x)}{p^-}} > \frac{\lambda}{4} \right\}} |D^2 u|^{\frac{\gamma_0 p(x)}{p^-}} dx \right. \\ &\qquad \left. + \frac{1}{\delta} \int_{B^+_{\tau_k}(y^k) \cap \left\{ |f|^{\frac{\gamma_0 p(x)}{p^-}} > \frac{\lambda \delta}{4} \right\}} |f|^{\frac{\gamma_0 p(x)}{p^-}} dx \right) \\ &\leq \frac{\epsilon c}{\lambda} \left(\int_{B^+_{s_2 \rho} \cap \left\{ |D^2 u|^{\frac{\gamma_0 p(x)}{p^-}} > \frac{\lambda}{4} \right\}} |D^2 u|^{\frac{\gamma_0 p(x)}{p^-}} dx \right. \\ &\qquad \left. + \frac{1}{\delta} \int_{B^+_{s_2 \rho} \cap \left\{ |f|^{\frac{\gamma_0 p(x)}{p^-}} > \frac{\lambda \delta}{4} \right\}} |f|^{\frac{\gamma_0 p(x)}{p^-}} dx \right). \end{aligned} \tag{2.3.39}$$

Now, we derive the main estimate (2.3.12). Using (2.3.32), a direct computation yields

$$\begin{aligned}\int_{B^+_{s_1 \rho}} |D^2 u|^{p(x)} dx &= \int_{B^+_{s_1 \rho}} \left(|D^2 u|^{\frac{\gamma_0 p(x)}{p^-}} \right)^{\frac{p^-}{\gamma_0}} dx \\ &= \int_0^{\infty} \frac{p^-}{\gamma_0} \lambda^{\frac{p^-}{\gamma_0} - 1} |E(\lambda)| d\lambda \\ &= \frac{p^-}{\gamma_0} K^{\frac{p^-}{\gamma_0} - 1} \int_0^{\infty} \lambda^{\frac{p^-}{\gamma_0} - 1} |E(K\lambda)| d(K\lambda) \\ &\leq \frac{p^-}{\gamma_0} K^{\frac{p^-}{\gamma_0}} \left(\int_0^{A\lambda_0} \lambda^{\frac{p^-}{\gamma_0} - 1} |E(K\lambda)| d\lambda + \int_{A\lambda_0}^{\infty} \lambda^{\frac{p^-}{\gamma_0} - 1} |E(K\lambda)| d\lambda \right) \\ &\leq K^{\frac{p^-}{\gamma_0}} (A\lambda_0)^{\frac{p^-}{\gamma_0}} \left| B^+_{s_1 \rho} \right| + \frac{p^-}{\gamma_0} K^{\frac{p^-}{\gamma_0}} \int_{A\lambda_0}^{\infty} \lambda^{\frac{p^-}{\gamma_0} - 1} |E(K\lambda)| d\lambda \\ &=: I_1 + I_2. \end{aligned} \tag{2.3.40}$$

We first estimate I_1. Recalling the definition of γ_0 in (2.3.16), we have

$$\frac{\gamma_0 p(x)}{p^-} \leq \gamma_1 (1 - \frac{\omega(4\rho_0)}{\gamma_1})(1 + \frac{\omega(4\rho)}{\gamma_1}) \leq \gamma_1.$$

From (2.3.14), (2.3.15) and the definitions of λ_0, A and K in (2.3.17), (2.3.19) and (2.3.35), we deduce

$$\begin{aligned}
I_1 &\leq c\frac{\left|B^+_{s_1\rho}\right|}{(s_2-s_1)^{\frac{np^-}{\gamma_0}}}\left\{\fint_{B^+_{2\rho}}\left[|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}+\frac{1}{\delta}\left(|f|^{\frac{\gamma_0 p(x)}{p^-}}+1\right)\right]dx\right\}^{\frac{p^-}{\gamma_0}}\\
&\leq \frac{c\left|B^+_{2\rho}\right|^{1-\frac{p^-}{\gamma_0}}}{(s_2-s_1)^{\frac{n\gamma_2}{\gamma_0}}}\left\{\int_{B^+_{2\rho}}\left(|D^2u|^{\gamma_1}+1\right)dx+\frac{1}{\delta}\int_{B^+_{2\rho}}\left(|f|^{p(x)}+1\right)dx\right\}^{\frac{p^-}{\gamma_0}}\\
&\leq \frac{c\left|B^+_{2\rho}\right|^{1-\frac{\gamma_2}{\gamma_0}}\left(1+\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}}}{(s_2-s_1)^{\frac{n\gamma_2}{\gamma_0}}}. \qquad (2.3.41)
\end{aligned}$$

On the other hand, we infer from (2.3.39) that

$$\begin{aligned}
I_2 &\leq c\int_{A\lambda_0}^{\infty}\lambda^{\frac{p^-}{\gamma_0}-2}\epsilon c\Bigg(\int_{B^+_{s_2\rho}\cap\left\{|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda}{4}\right\}}|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}dx\\
&\qquad\qquad +\frac{1}{\delta}\int_{B^+_{s_2\rho}\cap\left\{|f|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}}|f|^{\frac{\gamma_0 p(x)}{p^-}}dx\Bigg)d\lambda\\
&\leq c\epsilon\Bigg\{\int_0^\infty\left(\frac{\lambda}{4}\right)^{\frac{p^-}{\gamma_0}-2}\left(\int_{B^+_{s_2\rho}\cap\left\{|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda}{4}\right\}}|D^2u|^{\frac{\gamma_0 p(x)}{p^-}}dx\right)d\left(\frac{\lambda}{4}\right)\\
&+\left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}}\int_0^\infty\left(\frac{\lambda\delta}{4}\right)^{\frac{p^-}{\gamma_0}-2}\left(\int_{B^+_{s_2\rho}\cap\left\{|f|^{\frac{\gamma_0 p(x)}{p^-}}>\frac{\lambda\delta}{4}\right\}}|f|^{\frac{\gamma_0 p(x)}{p^-}}dx\right)d\left(\frac{\lambda\delta}{4}\right)\Bigg\}.
\end{aligned}$$

Then, applying (2.3.14) and (2.3.33), we obtain

$$\begin{aligned}
I_2 &\leq c\epsilon\left\{\int_{B^+_{s_2\rho}}|D^2u|^{p(x)}dx+\left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}}\int_{B^+_{s_2\rho}}|f|^{p(x)}dx\right\}\\
&\leq c_5\epsilon\int_{B^+_{s_2\rho}}|D^2u|^{p(x)}dx+c\epsilon\left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}}, \qquad (2.3.42)
\end{aligned}$$

for some $c_5=c_5(n,\Lambda,\gamma_1,\gamma_2,\omega(\cdot))>0$. Accordingly, we insert (2.3.41) and

(2.3.42) into (2.3.40) to discover

$$\int_{B^+_{s_1\rho}} |D^2u|^{p(x)}dx \le c_5\epsilon \int_{B^+_{s_2\rho}} |D^2u|^{p(x)}dx + \frac{c\left|B^+_{2\rho}\right|^{1-\frac{\gamma_2}{\gamma_0}}\left(1+\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}}}{(s_2-s_1)^{\frac{n\gamma_2}{\gamma_0}}} + \frac{c\epsilon}{\delta^{\frac{\gamma_2}{\gamma_0}}}.$$

Then, by taking $\epsilon = \epsilon(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 0$ small enough so that $0 < c_5\epsilon \le \frac{1}{2}$, from which $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 0$ is also selected, we finally obtain

$$\int_{B^+_{s_1\rho}} |D^2u|^{p(x)}dx \le \frac{1}{2}\int_{B^+_{s_2\rho}} |D^2u|^{p(x)}dx + \frac{c\left|B^+_{2\rho}\right|^{1-\frac{\gamma_2}{\gamma_0}}}{(s_2-s_1)^{\frac{n\gamma_2}{\gamma_0}}} + c.$$

Since s_1 and s_2 with $1 \le s_1 < s_2 \le 2$ are arbitrary, we apply Lemma 2.3.9 to find

$$\int_{B^+_{\rho}} |D^2u|^{p(x)}dx \le c\left(\left|B^+_{2\rho}\right|^{1-\frac{\gamma_2}{\gamma_0}} + 1\right) \le c_6\rho^{-\frac{n(\gamma_2-\gamma_1+\omega(4\rho_0))}{\gamma_1-\omega(4\rho_0)}} \tag{2.3.43}$$

for some $c_6 = c_6(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot)) > 1$, which, together with (2.3.3), implies the desired estimates (2.3.13). $\square$

2.3.3 Global $W^{2,p(\cdot)}$-estimates

We now prove our main result, Theorem 2.3.1, by using standard covering and flattening arguments, along with the a priori interior and boundary $W^{2,p(\cdot)}$-estimates, which were derived in the previous section. We first establish the global estimate under the a priori assumption that there exists a solution

$$u \in W^{2,p(\cdot)}(\Omega) \tag{2.3.44}$$

of (2.0.1), and remove it by an approximation procedure, in order to complete our proof. Hereafter, we denote by c to mean a universal constant being dependent only on $n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \partial\Omega, \mathrm{diam}(\Omega)$, and R.

Proof of Theorem 2.3.1. Let us first fix any point $x^0 = (x^{0\prime}, x^0_n) \in \partial\Omega$. From the assumption $\partial\Omega \in C^{1,1}$, there exist $r > 0$ and a $C^{1,1}$ function $\mu = \mu(x') : \mathbb{R}^{n-1} \to \mathbb{R}$, in a new coordinate system, still say x-coordinate system, such that $D_{x'}\mu(x^{0\prime}) = 0$ and $\|D^2_{x'}\mu\|_{L^\infty(\mathbb{R}^{n-1})} < \infty$, and

$$\Omega \cap B_r(x^0) = \{x \in \Omega : x_n > \mu(x')\} \cap B_r(x^0). \tag{2.3.45}$$

Note that the above condition (2.3.45) is also satisfied for all $\tilde{r} < r$ instead of r.

We first obtain a local estimate in $\Omega \cap B_r(x^0)$, for some sufficiently small $0 < r \leq R$, to be determined later, satisfying (2.3.45). In order to flatten out the boundary near x^0, we need to change coordinates. To deal with it, we define

$$\begin{cases} y_i &= x_i & =: \varphi^i(x), \quad \text{if } i = 1, 2, \cdots, n-1, \\ y_n &= x_n - \mu(x') & =: \varphi^n(x), \end{cases}$$

and write $y = \varphi(x)$. We further set $\psi := \varphi^{-1}$ and so $x = \psi(y)$. Next we define

$$\tilde{u}(y) = u(\psi(y)), \quad \tilde{p}(y) = p(\psi(y)),$$

$$\tilde{f}(y) = f(\psi(y)) - a_{ij}(\psi(y))\varphi^l_{x_i x_j}(\psi(y))D_{y_l}\tilde{u} \ \text{ and}$$

$$(\tilde{a}_{lm}(y)) = \tilde{\mathbf{A}}(y) = [D\varphi(\psi(y))] \cdot \mathbf{A}(\psi(y)) \cdot [D\varphi(\psi(y))]^t.$$

Note that $\tilde{\mathbf{A}}$ is uniformly elliptic with the ellipticity constant Λ and $\tilde{p}$ satisfies that $\gamma_1 \leq \tilde{p}(\cdot) \leq \gamma_2$ and

$$|\tilde{p}(y^2) - \tilde{p}(y^1)| \leq \omega(|\psi(y^2) - \psi(y^1)|) \leq \omega(\|D\psi\|_{L^\infty}|y^2 - y^1|) =: \tilde{\omega}(|y^2 - y^1|),$$

where $y^1, y^2 \in \mathbb{R}^n$ and $\tilde{\omega}(\rho) = \omega(\|D\psi\|_{L^\infty}\rho)$, and hence

$$\tilde{\omega}(\rho) \log\left(\frac{1}{\rho}\right) \leq \tilde{M}(\mu, \omega(\cdot)), \quad \text{for all } \rho \in (0, 1).$$

Take $\rho = \rho(\rho_0, r, \mu) > 0$ such that $\rho \leq \rho_0$ and $B^+_{4\rho} \subset \varphi(\Omega \cap B_r(x^0))$, where ρ_0 is given by (2.3.9). Then, by the a priori assumption (2.3.44) and the imposed conditions on f, $\mathbf{A}$ and $\partial\Omega$, we see that $\tilde{u}$ is in $W^{2,\tilde{p}(\cdot)}(B^+_{4\rho})$ and a solution of

$$\begin{cases} \tilde{a}_{lm} D_{y_l y_m}\tilde{u} &= \tilde{f} & \text{in} & B^+_{4\rho}, \\ \tilde{u} &= 0 & \text{on} & T_{4\rho}, \end{cases} \tag{2.3.46}$$

and $\tilde{f} \in L^{\tilde{p}(\cdot)}(B^+_{4\rho})$ with the estimate

$$\|\tilde{f}\|_{L^{\tilde{p}(\cdot)}(B^+_{4\rho})} \leq c(\mu)\left(\|f(\psi(y))\|_{L^{\tilde{p}(\cdot)}(B^+_{4\rho})} + \|D\tilde{u}\|_{L^{\tilde{p}(\cdot)}(B^+_{4\rho})}\right), \tag{2.3.47}$$

where $c(\mu)$ is a constant depending only on n, Λ and μ. Furthermore, we have

$$\begin{aligned}
[\tilde{\mathbf{A}}]_{4\rho} &\leq c(\mu)\left([\mathbf{A}]_R + \|D_{x'}\mu\|_{L^\infty(B'_r(x^{0\prime}))} + \|D_{x'}\mu\|^2_{L^\infty(B'_r(x^{0\prime}))}\right) \\
&\leq c(\mu)\left(\delta + \|D_{x'}\mu\|_{L^\infty(B'_r(x^{0\prime}))} + \|D_{x'}\mu\|^2_{L^\infty(B'_r(x^{0\prime}))}\right) \\
&\leq c(\mu)\left(\delta + r\|D^2_{x'}\mu\|_{L^\infty(B'_r(x^{0\prime}))} + r^2\|D^2_{x'}\mu\|^2_{L^\infty(B'_r(x^{0\prime}))}\right) \\
&\leq c(\mu)\left(\delta + r + r^2\right),
\end{aligned}$$

where we used the assumption $\partial\Omega \in C^{1,1}$ for the last inequality.

We notice from the above results that all the hypotheses of Theorem 2.3.4 are fulfilled with respect to the equation (2.3.46), by choosing $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \mu) > 0$ and $r = r(n, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), \mu) > 0$ sufficiently small. Consequently, Theorem 2.3.4 (ii) allows us to get

$$\begin{aligned}
\|D^2\tilde{u}\|_{L^{\tilde{p}(\cdot)}(B^+_\rho)} &\leq c\left(\|\tilde{f}\|_{L^{\tilde{p}(\cdot)}(B^+_{4\rho})} + \|\tilde{u}\|_{L^{\gamma_1}(B^+_{4\rho})}\right) \\
&\leq c\left(\|f(\psi(y))\|_{L^{\tilde{p}(\cdot)}(B^+_{4\rho})} + \|D\tilde{u}\|_{L^{\tilde{p}(\cdot)}(B^+_{4\rho})} + \|\tilde{u}\|_{L^{\gamma_1}(B^+_{4\rho})}\right),
\end{aligned}$$

where the last inequality follows from (2.3.47). Therefore, by converting back to the x-variables, we obtain

$$\begin{aligned}
\|D^2u\|_{L^{p(\cdot)}(V_{x^0})} &\leq c\left(\|f\|_{L^{p(\cdot)}(U_{x^0})} + \|Du\|_{L^{p(\cdot)}(U_{x^0})} + \|u\|_{L^{\gamma_1}(U_{x^0})}\right) \\
&\leq c\left(\|f\|_{L^{p(\cdot)}(\Omega)} + \|Du\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{\gamma_1}(\Omega)}\right), \quad (2.3.48)
\end{aligned}$$

where $V_{x^0} := \psi(B^+_\rho)$ and $U_{x^0} := \psi(B^+_{4\rho})$. Since $\partial\Omega$ is compact, it can be covered by a finite number of sets $V_{x^1}, V_{x^2}, \cdots, V_{x^N}$ for some points $x^j \in \partial\Omega$, $j = 1, 2, \cdots, N$, as above. Using a standard covering argument, we infer from Theorem 2.3.4 (i) that

$$\|D^2u\|_{L^{p(\cdot)}(V)} \leq c\left(\|f\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{\gamma_1}(\Omega)}\right), \quad (2.3.49)$$

for some open set $V \subset\subset \Omega$ so that $\Omega \subset V \cup \left(\bigcup_{j=1}^N V_{x^j}\right)$. By combining the estimates (2.3.48), when $x^0 = x^1, \ldots, x^N$, with (2.3.49), we conclude that

$$\|D^2u\|_{L^{p(\cdot)}(\Omega)} \leq c\left(\|f\|_{L^{p(\cdot)}(\Omega)} + \|Du\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{\gamma_1}(\Omega)}\right).$$

From [38], we recall the interpolation inequality for the variable exponent spaces that for any $\eta \in (0,1)$,

$$\|Du\|_{L^{p(\cdot)}(\Omega)} \le \eta \|D^2 u\|_{L^{p(\cdot)}(\Omega)} + c(\eta)\|u\|_{L^{p(\cdot)}(\Omega)},$$

from which we deduce

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \le c\left(\|f\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_{L^{\gamma_1}(\Omega)}\right). \tag{2.3.50}$$

Furthermore, by the uniqueness of solutions for a homogeneous equation, we finally obtain the desired estimates

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \le c\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{2.3.51}$$

Indeed, to do this, we argue by contradiction. Suppose that (2.3.51) is not true. Then there exist $\{u_l\}_{l=1}^{\infty}$ and $\{f_l\}_{l=1}^{\infty}$ such that u_l is a solution of

$$\left\{\begin{array}{rcll} a_{ij}D_{ij}u_l &=& f_l & \text{in} \quad \Omega, \\ u_l &=& 0 & \text{on} \quad \partial\Omega, \end{array}\right.$$

with

$$\|u_l\|_{W^{2,p(\cdot)}(\Omega)} > l\|f_l\|_{L^{p(\cdot)}(\Omega)}, \tag{2.3.52}$$

for any $l \ge 1$. Without loss of generality, we may assume

$$\|u_l\|_{W^{2,p(\cdot)}(\Omega)} = 1,$$

and then (2.3.52) implies $\|f_l\|_{L^{p(\cdot)}(\Omega)} < \frac{1}{l}$. Then there exist a subsequence of $\{u_l\}_{l=1}^{\infty}$, which we still denote by $\{u_l\}_{l=1}^{\infty}$, and a function $u_0 \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ such that

$$u_l \rightharpoonup u_0 \text{ in } W^{2,p(\cdot)}(\Omega) \quad \text{as} \quad l \to \infty.$$

Since $W^{1,p(\cdot)}(\Omega)$ is compactly embedded in $L^{p(\cdot)}(\Omega)$, see [28], we may assume that u_l converges strongly to u_0 in $L^{p(\cdot)}(\Omega)$. Then, it is easy to check that $u_0 \in W^{2,p(\cdot)}(\Omega) \subset W^{2,\gamma_1}(\Omega)$ is a solution of

$$\left\{\begin{array}{rcll} a_{ij}D_{ij}u_0 &=& 0 & \text{in} \quad \Omega, \\ u_0 &=& 0 & \text{on} \quad \partial\Omega. \end{array}\right. \tag{2.3.53}$$

By the uniqueness for solutions of (2.3.53), see [25], it is clear that $u_0 = 0$

in Ω. However, it follows from (2.3.50) that

$$1 \leq c\left(\|f_l\|_{L^{p(\cdot)}(\Omega)} + \|u_l\|_{L^{p(\cdot)}(\Omega)} + \|u_l\|_{L^{\gamma_1}(\Omega)}\right) \to 0 \quad \text{as} \quad l \to \infty,$$

which is a contradiction.

We now need only to remove the a priori assumption (2.3.44). To do this, choose a sequence $\{\mathbf{A}^l\}_{l=1}^{\infty} = \{(a_{ij}^l)\}_{l=1}^{\infty}$ of smooth matrix functions with uniform (δ, R)-vanishing property such that

$$a_{ij}^l \to a_{ij} \ \text{ in } L^t(\Omega) \quad \text{for each } 1 < t < \infty. \tag{2.3.54}$$

We also select a sequence $\{f_l\}_{l=1}^{\infty}$ of smooth functions in $C_0^{\infty}(\Omega)$ satisfying

$$f_l \to f \ \text{ in } \ L^{p(\cdot)}(\Omega) \quad \text{and} \quad \|f_l\|_{L^{p(\cdot)}(\Omega)} \leq \|f\|_{L^{p(\cdot)}(\Omega)} + 1. \tag{2.3.55}$$

We observe from [25] that there exists the unique solution $u_l \in W^{2,\gamma_2}(\Omega) \cap W_0^{1,\gamma_2}(\Omega)$ of

$$\begin{cases} a_{ij}^l D_{ij} u_l &= f_l \quad \text{in} \quad \Omega, \\ u_l &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.3.56}$$

Therefore, we have $u_l \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$. Then (2.3.51) implies

$$\|u_l\|_{W^{2,p(\cdot)}(\Omega)} \leq c\|f_l\|_{L^{p(\cdot)}(\Omega)} \leq c\left(\|f\|_{L^{p(\cdot)}(\Omega)} + 1\right), \tag{2.3.57}$$

which means that $\{u_l\}_{l=1}^{\infty}$ is uniformly bounded in $W^{2,p(\cdot)}(\Omega)$. Therefore there exist a subsequence of $\{u_l\}_{l=1}^{\infty}$, which is still denoted by $\{u_l\}_{l=1}^{\infty}$, and a function $u \in W^{2,p(\cdot)}(\Omega)$ such that

$$u_l \rightharpoonup u \ \text{ weakly in } W^{2,p(\cdot)}(\Omega). \tag{2.3.58}$$

In view of (2.3.54)-(2.3.56) and (2.3.58), it is easy to check that $u \in W^{2,p(\cdot)}(\Omega)$ is a solution of (2.0.1), and hence we can remove the a priori assumption (2.3.44).

The uniqueness for solutions of the problem (2.0.1) follows from the linearity of (2.0.1) and the uniqueness for solutions of (2.3.53). Consequently, we complete the proof of Theorem 2.3.1. □

Chapter 3

Regularity theory for nondivergence parabolic equations

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and set $\Omega_T := \Omega \times (0,T]$ for the cylinder in $\mathbb{R}^{n+1}$ with base Ω and height T. We consider the following Dirichlet problem for the second order parabolic equation in nondivergence form:

$$\begin{cases} u_t - a_{ij}(x,t)D_{ij}u(x,t) = f(x,t) & \text{in} \quad \Omega_T, \\ u(x,t) = 0 & \text{on} \quad \partial_p\Omega_T, \end{cases} \tag{3.0.1}$$

where $\partial_p\Omega_T := (\partial\Omega \times [0,T]) \cup (\Omega \times \{t=0\})$ is the parabolic boundary of Ω_T and matrix $\mathbf{A} = (a_{ij})$, which is composed of coefficients, is assumed to be symmetric and satisfy the uniform parabolicity condition, i.e., there exists a positive constant Λ, called the parabolicity constant, such that

$$\Lambda^{-1}|\eta|^2 \leq \langle \mathbf{A}(z)\eta, \eta\rangle \leq \Lambda|\eta|^2 \tag{3.0.2}$$

for all $\eta \in \mathbb{R}^n$ and a.e. $z = (x,t) \in \mathbb{R}^{n+1}$. The aim of this chapter is twofold. One is to derive global weighted Orlicz estimates for the problem (3.0.1), and the other is to prove the global Calderón-Zygmund estimates for the problem (3.0.1) in weighted variable exponent Lebesgue spaces.

We introduce some standard notation and definitions that will be used throughout the chapter. The variable in $\mathbb{R}^{n+1}$ is termed $z = (x,t)$ for the spatial variables $x = (x', x_n) = (x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n$ and the time variable $t \in \mathbb{R}$. For a function $g : U \subset \mathbb{R}^{n+1} \to \mathbb{R}$, we denote the spatial gradient of g by $Dg = (D_1 g, \cdots, D_n g)$, the spatial Hessian of g by $D^2 g = (D_{ij} g)$, where $D_i g = D_{x_i} g = \frac{\partial g}{\partial x_i}$, $D_{ij} g = D_{x_i x_j} g = \frac{\partial^2 g}{\partial x_i \partial x_j}$ for $i, j = 1, \cdots, n$, while the time derivative of g by $g_t = D_t g = \frac{\partial g}{\partial t}$. As usual, the parabolic distance d_p between two points $\xi = (y, s), \tilde{\xi} = (\tilde{y}, \tilde{s}) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ is denoted by

$$d_p(\xi, \tilde{\xi}) := \max \left\{ |y - \tilde{y}|, \sqrt{|s - \tilde{s}|} \right\},$$

where $|\cdot|$ is the Euclidean norm. In this chapter, we shall use a parabolic cylinder of the form

$$Q_r(\xi) = Q_r(y, s) := B_r(y) \times \left(s - r^2, s + r^2\right)$$

with center $\xi = (y, s) \in \mathbb{R}^{n+1}$ and radius $r > 0$, where $B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}$ is the open ball in $\mathbb{R}^n$ with center y and radius r. Its parabolic boundary is denoted by

$$\partial_p Q_r(\xi) = \partial_p Q_r(y, s) = \left(\partial B_r(y) \times \left[s - r^2, s + r^2\right)\right) \cup \left(B_r(y) \times \left\{t = s - r^2\right\}\right).$$

For the sake of simplicity, we abbreviate $B_r^+(y) = B_r(y) \cap \{x_n > 0\}$, $B_r = B_r(0)$ and $B_r^+ = B_r^+(0)$. We also write $Q_r = B_r \times \left(-r^2, r^2\right)$, $Q_r^+ = B_r^+ \times \left(-r^2, r^2\right)$. In our further considerations, we shall use the notations $T_r = Q_r \cap \{x_n = 0\}$ and $T_r(y, s) = T_r + (y, s)$. Furthermore, we shall employ a parabolic cube of the form

$$C_r(\xi) = C_r(y, s) := \{x \in \mathbb{R}^n : |x_i - y_i| < r,\ i = 1, \ldots, n\} \times (s - r^2, s + r^2)$$

for $\xi = (y, s) \in \mathbb{R}^{n+1}$ and $r > 0$.

For a vector valued function $\mathbf{f} : U \to \mathbb{R}^N$, $N \geq 1$, we denote $\bar{\mathbf{f}}_U$ by the integral average of $\mathbf{f}$ on U, that is,

$$\bar{\mathbf{f}}_U = \fint_U \mathbf{f}(z) dz = \frac{1}{|U|} \int_U \mathbf{f}(z) dz.$$

The following is our principal assumption on the coefficient matrix $\mathbf{A}$.

Definition 3.0.10. We say that $\mathbf{A}$ is (δ, R)-*vanishing* if

$$\sup_{0<r\leq R} \sup_{\xi \in \mathbb{R}^{n+1}} \left(\fint_{Q_r(\xi)} \left| \mathbf{A}(z) - \overline{\mathbf{A}}_{Q_r(\xi)} \right|^2 dz \right)^{\frac{1}{2}} \leq \delta. \tag{3.0.3}$$

We remark that scaling the given equation allows R in the above definition, to be any positive number larger than 1, while δ is the scaling invariant. Note that coefficients a_{ij} can be extended from Ω_T to $\mathbb{R}^{n+1}$, preserving condition (3.0.3); see [4]. Therefore, coefficients a_{ij} are defined in $\mathbb{R}^{n+1}$ throughout this chapter.

A locally integrable function f is *of bounded mean oscillation* (BMO) on $\mathbb{R}^{n+1}$, denoted by $f \in \mathrm{BMO}(\mathbb{R}^{n+1})$, if the BMO seminorm

$$\|f\|_* := \sup_{Q \subset \mathbb{R}^{n+1}} \fint_Q \left| f - \overline{f}_Q \right| \, dxdt$$

is finite, where the supremum is taken over all parabolic cylinders Q in $\mathbb{R}^{n+1}$. In this chapter, we always assume that $\mathbf{A} = (a_{ij})$ is in the BMO space of functions with small BMO seminorms, which was defined in (3.0.3). This condition is weaker than the VMO condition, which was assumed in [9]. In addition, condition (3.0.3) is equivalent to the small BMO condition $\|\mathbf{A}\|_* \leq \delta$, using the John-Nirenberg inequality; see [42]. This way, we utilize

$$[\mathbf{A}]_R := \sup_{0<r\leq R} \sup_{\xi \in \mathbb{R}^{n+1}} \fint_{Q_r(\xi)} \left|\mathbf{A}(z) - \overline{\mathbf{A}}_{Q_r(\xi)}\right| dz \leq \delta. \tag{3.0.4}$$

as the definition of (δ, R)-*vanishing* of the coefficient matrix $\mathbf{A}$.

Note that Bramanti and Cerutti [9] showed that if $f \in L^p(\Omega_T)$ for any constant p with $1 < p < \infty$, there exists a unique *strong solution* u, i.e., a function $u \in W^{2,1}_p(\Omega_T)$ which satisfies the equation (3.0.1) almost everywhere in Ω_T and $u \equiv 0$ on $\partial_p \Omega_T$ in the trace sense, whose coefficient matrix belongs to the class of functions of VMO type. This result can be naturally extended to the same equation whose coefficient matrix is (δ, R)-vanishing for some sufficiently small $\delta > 0$ and any $R > 0$.

3.1 Preliminary results

We start this chapter with recalling the interior and boundary *a priori* $W^{2,1}_q$-estimates and the global $W^{2,1}_q$-estimates in a $C^{1,1}$ domain that have been proved in [9].

Lemma 3.1.1. *Let $1 < q < \infty$. There exist a small $\delta = \delta(\Lambda, n, q) > 0$ and $c = c(\Lambda, n, q) > 0$ such that the following hold for any fixed $r > 0$:*

(i) (Interior estimates) If $\mathbf{A}$ is $(\delta, 2r)$-vanishing and $f \in L^q(Q_{2r})$, then

for any strong solution $u \in W_q^{2,1}(Q_{2r})$ *of*

$$u_t - a_{ij}D_{ij}u = f \quad \textit{in} \quad Q_{2r},$$

we have the estimate

$$\|u_t\|_{L^q(Q_r)} + \|D^2u\|_{L^q(Q_r)} \le c\left(\|f\|_{L^q(Q_{2r})} + \frac{1}{r^2}\|u\|_{L^q(Q_{2r})}\right).$$

(ii) *(Boundary estimates) If* $\mathbf{A}$ *is* $(\delta, 2r)$*-vanishing and* $f \in L^q(Q_{2r}^+)$*, then for any strong solution* $u \in W_q^{2,1}(Q_{2r}^+)$ *of*

$$\begin{cases} u_t - a_{ij}D_{ij}u &= f \quad \textit{in} \quad Q_{2r}^+, \\ u &= 0 \quad \textit{on} \quad T_{2r}, \end{cases}$$

we have the estimate

$$\|u_t\|_{L^q(Q_r^+)} + \|D^2u\|_{L^q(Q_r^+)} \le c\left(\|f\|_{L^q(Q_{2r}^+)} + \frac{1}{r^2}\|u\|_{L^q(Q_{2r}^+)}\right).$$

Moreover, let $\partial\Omega \in C^{1,1}$*. Then there exists a small* $\delta = \delta(\Lambda, n, q, \partial\Omega) > 0$ *such that if* $f \in L^q(\Omega_T)$ *and* $\mathbf{A}$ *is* (δ, R)*-vanishing for some* $R > 0$*, then the problem (3.0.1) has a unique strong solution* $u \in \overset{\circ}{W}{}_q^{2,1}(\Omega_T)$*, and we have the estimate*

$$\|u\|_{W_q^{2,1}(\Omega_T)} \le c\|f\|_{L^q(\Omega_T)}, \tag{3.1.1}$$

for some $c = c(n, \Lambda, q, \Omega, R, T) > 0$.

We next derive the comparison estimates in L^q spaces with $1 < q < \infty$ by using a compactness argument. These estimates play crucial roles in the proofs of local weighted Orlicz estimates as well as those of local $W_{p(\cdot)}^{2,1}$-estimates in Chapters 3.2.3 and 3.3.3. In what follows, we denote by c any positive constant depending only on n, Λ and q, which may vary from line to line.

We first prove the Poincaré type inequalities in Sobolev space $W_q^{2,1}$, which will be used in the derivation of the subsequent lemmas.

Lemma 3.1.2. *For any* $1 < q < \infty$*, let* $h \in W_q^{2,1}(Q_4)$*. Then there is a positive constant* c *depending only on* q *and* n *so that*

$$\fint_{Q_4} |h - \overline{h}_{Q_4} - \overline{(Dh)}_{Q_4} \cdot x|^q dz \le c \fint_{Q_4} \left(|h_t|^q + |D^2h|^q\right) dz. \tag{3.1.2}$$

Proof. We argue by contradiction. Suppose that (3.1.2) is not true. Then there exists a sequence $\{h_k\}_{k=1}^{\infty}$ in $W_q^{2,1}(Q_4)$ satisfying

$$\fint_{Q_4} |h_k - \overline{h_k}_{Q_4} - \overline{(Dh_k)}_{Q_4} \cdot x|^q dz > k \fint_{Q_4} \left(|(h_k)_t|^q + |D^2 h_k|^q\right) dz. \tag{3.1.3}$$

By the normalization, we may assume

$$\fint_{Q_4} |h_k - \overline{h_k}_{Q_4} - \overline{(Dh_k)}_{Q_4} \cdot x|^q dz = 1,$$

and then the inequality (3.1.3) implies

$$\fint_{Q_4} \left(|(h_k)_t|^q + |D^2 h_k|^q\right) dz < \frac{1}{k}.$$

Now let us consider $\tilde{h}_k := h_k - \overline{h_k}_{Q_4} - \overline{(Dh_k)}_{Q_4} \cdot x$. Then it is easy to check that

$$\fint_{Q_4} \tilde{h}_k dz = \fint_{Q_4} D\tilde{h}_k dz = 0, \tag{3.1.4}$$

$$\fint_{Q_4} |\tilde{h}_k|^q dz = 1, \text{ and } \fint_{Q_4} \left(|(\tilde{h}_k)_t|^q + |D^2 \tilde{h}_k|^q\right) dz < \frac{1}{k} \le 1. \tag{3.1.5}$$

In addition, we use the interpolation inequality (see [5, Theorem 5.2]) for each time slice $B_4 \times \{t\}$ with $t \in [-4^2, 4^2]$, to obtain

$$\begin{aligned} \int_{Q_4} |D\tilde{h}_k|^q dz &\le \int_{[-4^2,4^2]} c \left(\int_{B_4} |\tilde{h}_k|^q dx + \int_{B_4} |D^2 \tilde{h}_k|^q dx \right) dt \\ &= c \left(\int_{Q_4} |\tilde{h}_k|^q dz + \int_{Q_4} |D^2 \tilde{h}_k|^q dz \right), \end{aligned} \tag{3.1.6}$$

and in turn, it follows from (3.1.5) that

$$\int_{Q_4} |D\tilde{h}_k|^q dz \le c. \tag{3.1.7}$$

In view of (3.1.5) and (3.1.7), we then see that $\{\tilde{h}_k\}_{k=1}^{\infty}$ is bounded in $W_q^{2,1}(Q_4)$. Therefore there exist a subsequence of $\{\tilde{h}_k\}_{k=1}^{\infty}$, which we still denote by $\{\tilde{h}_k\}_{k=1}^{\infty}$, and a function $\tilde{h} \in W_q^{2,1}(Q_4)$ such that

$$\begin{cases} \tilde{h}_k \rightharpoonup \tilde{h} & \text{weakly in } W_q^{2,1}(Q_4), \\ \tilde{h}_k \to \tilde{h} & \text{strongly in } L^q(Q_4) \end{cases} \quad \text{as} \quad k \to \infty.$$

Then we infer from (3.1.5) that

$$\fint_{Q_4} |\tilde{h}|^q dz = 1 \quad \text{and} \quad \tilde{h}_t = D^2 \tilde{h} = 0. \tag{3.1.8}$$

So we can write $\tilde{h} = c_1 \cdot x + c_2$ for some constants $c_1 \in \mathbb{R}^n$ and $c_2 \in \mathbb{R}$. However, it follows from (3.1.4) that

$$c_1 = \fint_{Q_4} D\tilde{h} dz = 0 \quad \text{and} \quad c_2 = \fint_{Q_4} \tilde{h} dz = 0.$$

In turn, we see $\tilde{h} = 0$ in Q_4, which is a contradiction to the first equality in (3.1.8). □

Lemma 3.1.3. *For any $1 < q < \infty$, let $h \in W_q^{2,1}(Q_4^+)$ with $h = 0$ on T_4. Then there is a constant c depending only on q and n so that*

$$\fint_{Q_4^+} |h - \overline{(D_n h)}_{Q_4^+} x_n|^q dz \le c \fint_{Q_4^+} \left(|h_t|^q + |D^2 h|^q \right) dz. \tag{3.1.9}$$

Proof. Suppose that (3.1.9) is not true, then there exists a sequence $\{h_k\}_{k=1}^{\infty}$ in $W_q^{2,1}(Q_4^+)$ with $h_k = 0$ on T_4 such that

$$\fint_{Q_4^+} |h_k - \overline{(D_n h_k)}_{Q_4^+} x_n|^q dz > k \fint_{Q_4^+} \left(|(h_k)_t|^q + |D^2 h_k|^q \right) dz. \tag{3.1.10}$$

By the normalization, we may assume

$$\fint_{Q_4^+} |h_k - \overline{(D_n h_k)}_{Q_4^+} x_n|^q dz = 1.$$

Then the inequality (3.1.10) yields

$$\fint_{Q_4^+} \left(|(h_k)_t|^q + |D^2 h_k|^q \right) dz < \frac{1}{k}.$$

Setting $\tilde{h}_k := h_k - \overline{(D_n h_k)}_{Q_4^+} x_n$, we then easily see that

$$\fint_{Q_4^+} |\tilde{h}_k|^q dz = 1 \quad \text{and} \quad \fint_{Q_4^+} \left(|(\tilde{h}_k)_t|^q + |D^2 \tilde{h}_k|^q \right) dz < \frac{1}{k} \le 1. \tag{3.1.11}$$

In an analogous way that (3.1.7) has been deduced, we can infer from (3.1.11) instead of (3.1.4), that

$$\int_{Q_4^+} |D\tilde{h}_k|^q dz \leq c. \tag{3.1.12}$$

We also know that

$$\fint_{Q_4^+} D_n\tilde{h}_k dz = \fint_{Q_4^+} \left(D_n h_k - \overline{(D_n h_k)}_{Q_4^+} \right) dz = 0. \tag{3.1.13}$$

From (3.1.11) and (3.1.12), we see that $\{\tilde{h}_k\}_{k=1}^{\infty}$ is bounded in $W_q^{2,1}(Q_4^+)$, and so there exist a subsequence of $\{\tilde{h}_k\}_{k=1}^{\infty}$, which we still denote by $\{\tilde{h}_k\}_{k=1}^{\infty}$, and a function $\tilde{h} \in W_q^{2,1}(Q_4^+)$ with $\tilde{h} = 0$ on T_4 such that

$$\begin{cases} \tilde{h}_k \rightharpoonup \tilde{h} & \text{weakly in } W_q^{2,1}(Q_4^+), \\ \tilde{h}_k \to \tilde{h} & \text{strongly in } L^q(Q_4^+) \end{cases} \quad \text{as} \quad k \to \infty.$$

For $1 \leq i \leq n-1$, since $D_i\tilde{h}_k = 0$ on T_4, we apply the standard Poincaréé inequality for each time slice of Q_4^+, in order to discover that

$$\fint_{Q_4^+} \left| D_i\tilde{h}_k \right|^q dz \leq c \fint_{Q_4^+} |D^2\tilde{h}_k|^q dz = c \fint_{Q_4^+} |D^2 h_k|^q dz < \frac{c}{k} \longrightarrow 0$$

as $k \to \infty$, which implies that

$$D_i\tilde{h} = 0.$$

Furthermore, it is easy to check from (3.1.11) that

$$\fint_{Q_4^+} |\tilde{h}|^q dz = 1 \quad \text{and} \quad \tilde{h}_t = D^2\tilde{h} = 0. \tag{3.1.14}$$

So we can write $\tilde{h} = c_1 x_n + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$. However, since $\tilde{h} = 0$ on T_4, we have $c_2 = 0$, and then by (3.1.13) we see

$$c_1 = \fint_{Q_4^+} D_n\tilde{h} dz = 0.$$

Therefore, we finally have $\tilde{h} = 0$ in Q_4^+, which is a contradiction to the first equality in (3.1.14). This completes the proof. $\square$

Let us now derive the following comparison estimates.

Lemma 3.1.4. *Let $1 < q < \infty$. Assume that $\mathbf{B} = (b_{ij}) : \mathbb{R}^{n+1} \to \mathbb{R}^{n \times n}$*

satisfies the uniform parabolicity condition (3.0.2). *For any* $\epsilon \in (0,1)$, *there is* $\delta = \delta(\epsilon, n, \Lambda, q) > 0$ *such that the following hold:*

If $\mathbf{B}$ *is* $(\delta, 4)$*-vanishing and* $h \in W_q^{2,1}(Q_4)$ *is a solution of*

$$h_t - b_{ij} D_{ij} h = g \quad \text{in } Q_4 \tag{3.1.15}$$

satisfying

$$\fint_{Q_4} \left(|h_t|^q + |D^2 h|^q\right) dz \le 1 \quad \text{and} \quad \fint_{Q_4} |g|^q dz \le \delta,$$

then there exist a constant matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ *with* $\|\overline{\mathbf{B}}_{Q_4} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \le \epsilon$ *and a solution* $v \in W_q^{2,1}(Q_4)$ *of*

$$v_t - \tilde{b}_{ij} D_{ij} v = 0 \quad \text{in } Q_4 \tag{3.1.16}$$

satisfying

$$\fint_{Q_4} \left(|v_t|^q + |D^2 v|^q\right) dz \le 1, \tag{3.1.17}$$

and

$$\int_{Q_4} |h - \overline{h}_{Q_4} - (\overline{Dh})_{Q_4} \cdot x - v|^q dz \le \epsilon.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $h_l \in W_q^{2,1}(Q_4)$, $g_l \in L^q(Q_4)$ and $\mathbf{B}_l = (b_{ij}^l) : \mathbb{R}^{n+1} \to \mathbb{R}^{n \times n}$, where $l = 1, 2, \ldots$, such that $\mathbf{B}_l$ is uniformly parabolic with the parabolicity constant Λ satisfying $[\mathbf{B}_l]_4 \le \frac{1}{l}$, which implies that

$$\fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{l Q_4}| dz \le \frac{1}{l}, \tag{3.1.18}$$

and $h_l \in W_q^{2,1}(Q_4)$ is a solution of

$$(h_l)_t - b_{ij}^l D_{ij} h_l = g_l \quad \text{in } Q_4,$$

satisfying

$$\fint_{Q_4} \left(|(h_l)_t|^q + |D^2 h_l|^q\right) dz \le 1 \text{ and } \fint_{Q_4} |g_l|^q dz \le \frac{1}{l}, \tag{3.1.19}$$

but

$$\int_{Q_4} |h_l - \overline{h_l}_{Q_4} - (\overline{Dh_l})_{Q_4} \cdot x - v|^q dz > \epsilon_0, \tag{3.1.20}$$

for any constant matrix $\tilde{\mathbf{B}}$ with $\|\overline{\mathbf{B}}_{Q_4} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \leq \epsilon_0$ and any solution $v \in W_q^{2,1}(Q_4)$ of (3.1.16) with (3.1.17).

By virtue of the uniform parabolicity on $\mathbf{B}_l$ and (3.1.18), we infer

$$\fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{lQ_4}|^{q'} dz \leq (2\Lambda)^{q'-1} \fint_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{lQ_4}| dz \leq \frac{(2\Lambda)^{q'-1}}{l},$$

where $q' = \frac{q}{q-1}$. On the other hand, it is clear that $\{\overline{\mathbf{B}}_{lQ_4}\}_{l=1}^\infty$ is bounded in $\mathbb{R}^{n\times n}$, and so it has a subsequence, which is still denoted by $\{\overline{\mathbf{B}}_{lQ_4}\}$, such that

$$\overline{\mathbf{B}}_{lQ_4} \longrightarrow \mathbf{B}_0 \quad \text{in} \quad \mathbb{R}^{n\times n} \quad \text{as} \quad l \to \infty, \tag{3.1.21}$$

for some constant matrix $\mathbf{B}_0 = (b_{ij}^0)$. Therefore it follows

$$\mathbf{B}_l \longrightarrow \mathbf{B}_0 \quad \text{in} \quad L^{q'}(Q_4) \quad \text{as} \quad l \to \infty. \tag{3.1.22}$$

Let us now consider $v_l := h_l - \overline{h}_{lQ_4} - (\overline{Dh_l})_{Q_4} \cdot x$. Then we see from Lemma 3.1.2 that

$$\fint_{Q_4} |v_l|^q dz \leq c \fint_{Q_4} \left(|(v_l)_t|^q + |D^2 v_l|^q\right) dz = c \fint_{Q_4} \left(|(h_l)_t|^q + |D^2 h_l|^q\right) dz \leq c, \tag{3.1.23}$$

where the last inequality follows from (3.1.19). Moreover, in an analogous way to (3.1.6), the interpolation inequality leads us to get

$$\int_{Q_4} |Dv_l|^q dz \leq c \left(\int_{Q_4} |v_l|^q dz + \int_{Q_4} |D^2 v_l|^q dz \right),$$

and then it follows from (3.1.23) that

$$\int_{Q_4} |Dv_l|^q dz \leq c. \tag{3.1.24}$$

Therefore, in view of (3.1.23) and (3.1.24), we see that $\{v_l\}_{l=1}^\infty$ is bounded in $W_q^{2,1}(Q_4)$ and so there exist a subsequence of $\{v_l\}_{l=1}^\infty$, which is still denoted by $\{v_l\}_{l=1}^\infty$, and a function $v_0 \in W_q^{2,1}(Q_4)$ such that

$$\begin{cases} v_l \rightharpoonup v_0 & \text{weakly in } W_q^{2,1}(Q_4), \\ v_l \to v_0 & \text{strongly in } L^q(Q_4), \end{cases} \quad \text{as} \quad l \to \infty. \tag{3.1.25}$$

From (3.1.19), (3.1.22) and (3.1.25), we observe that $v_0 \in W_q^{2,1}(Q_4)$ is a

solution of

$$(v_0)_t - b^0_{ij} D_{ij} v_0 = 0 \quad \text{in} \;\; Q_4,$$

satisfying

$$⨍_{Q_4} \left(|(v_0)_t|^q + |D^2 v_0|^q \right) dz \le \liminf_{l\to\infty} ⨍_{Q_4} \left(|(v_l)_t|^q + |D^2 v_l|^q \right) dz \le 1. \tag{3.1.26}$$

However, it is a contradiction to (3.1.20). This completes the proof. □

Corollary 3.1.5. *Under the hypotheses and conclusion of Lemma 3.1.4, we have*

$$⨍_{Q_1} \left(|(h-v)_t|^q + |D^2(h-v)|^q \right) dz \le \epsilon.$$

Proof. From the assumptions of Lemma 3.1.4, we see

$$⨍_{Q_4} |g|^q dz \le \delta \quad \text{and} \quad ⨍_{Q_4} |\mathbf{B} - \overline{\mathbf{B}}_{Q_4}| dz \le \delta. \tag{3.1.27}$$

Apply Lemma 3.1.4 with any $\kappa > 0$ in place of ϵ in order to find a constant matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ with $\|\overline{\mathbf{B}}_{Q_4} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \le \kappa$ and a solution $v \in W^{2,q}(Q_4)$ of (3.1.16) such that

$$⨍_{Q_4} \left(|v_t|^q + |D^2 v|^q \right) dz \le 1 \text{ and } \int_{Q_4} |h - \overline{h}_{Q_4} - (\overline{Dh})_{Q_4} \cdot x - v|^q dz \le \kappa, \tag{3.1.28}$$

by taking $\delta = \delta(\kappa, n, \Lambda, q) > 0$ sufficiently small. Then we use the local estimates on derivatives of solutions to the equation (3.1.16) (see Theorem 9 in [35, page 61]) to have

$$\|v_t\|^q_{L^\infty(Q_2)} + \|D^2 v\|^q_{L^\infty(Q_2)} \le c ⨍_{Q_4} \left(|v_t|^q + |D^2 v|^q \right) dz \le c. \tag{3.1.29}$$

Setting $\tilde{h} := h - \overline{h}_{Q_4} - (\overline{Dh})_{Q_4} \cdot x - v$, one can readily see that $\tilde{h} \in W^{2,1}_q(Q_4)$ is a solution of

$$\tilde{h}_t - b_{ij} D_{ij} \tilde{h} = g + (b_{ij} - \tilde{b}_{ij}) D_{ij} v \quad \text{in } Q_4.$$

Then Lemma 3.1.1 gives

$$⨍_{Q_1} \left(|\tilde{h}_t|^q + |D^2 \tilde{h}|^q \right) dz \le c \left(⨍_{Q_2} \left| g + (b_{ij} - \tilde{b}_{ij}) D_{ij} v \right|^q dz + ⨍_{Q_2} |\tilde{h}|^q dz \right), \tag{3.1.30}$$

if we take $\delta = \delta(\kappa, n, \Lambda, q) > 0$ sufficiently small.

In view of (3.1.27) -(3.1.30) and (3.0.2), we consequently deduce

$$
\begin{aligned}
&\fint_{Q_1} \left(|(h-v)_t|^q + \left|D^2(h-v)\right|^q\right) dz = \fint_{Q_1} \left(|\tilde{h}_t|^q + |D^2\tilde{h}|^q\right) dz \\
&\le c\left(\fint_{Q_2} \left|g + (b_{ij} - \tilde{b}_{ij})D_{ij}v\right|^q dz + \fint_{Q_2} |\tilde{h}|^q dz\right) \\
&\le c\left(\fint_{Q_4} |g|^q dz + \|D^2 v\|^q_{L^\infty(Q_2)} \fint_{Q_4} \left|\mathbf{B} - \tilde{\mathbf{B}}\right|^q dz + \kappa\right) \\
&\le c\left(\fint_{Q_4} |g|^q dz + 2^{q-1} \fint_{Q_4} \left(\left|\mathbf{B} - \overline{\mathbf{B}}_{Q_4}\right|^q + \left|\overline{\mathbf{B}}_{Q_4} - \tilde{\mathbf{B}}\right|^q\right) dz + \kappa\right) \\
&\le c\left(\delta + (4\Lambda)^{q-1} \fint_{Q_4} \left|\mathbf{B} - \overline{\mathbf{B}}_{Q_4}\right| dz + 2^{q-1}\kappa^q + \kappa\right) \\
&\le c(\delta + \kappa),
\end{aligned}
$$

where the elementary inequality $(a+b)^\beta \le 2^{\beta-1}(a^\beta + b^\beta)$ for any $a, b > 0$ and $\beta \ge 1$ has been used in the third inequality. Hence, the proof is completed by choosing $\kappa = \kappa(\epsilon, n, \Lambda, q) > 0$ and $\delta = \delta(\epsilon, n, \Lambda, q) > 0$ small enough so that $c(\delta + \kappa) < \epsilon$. $\square$

The following is the flat boundary version of Lemma 3.1.4, which will be proved by the same argument as in Lemma 3.1.4 with Lemma 3.1.3 instead of Lemma 3.1.2.

Lemma 3.1.6. *Let $1 < q < \infty$. Assume that $\mathbf{B} = (b_{ij}) : \mathbb{R}^{n+1} \to \mathbb{R}^{n\times n}$ satisfies the uniform parabolicity condition (3.0.2). For any $\epsilon \in (0,1)$, there is $\delta = \delta(\epsilon, n, \Lambda, q) > 0$ such that the following hold:*

If $\mathbf{B}$ is $(\delta, 4)$-vanishing and $h \in W^{2,1}_q(Q_4^+)$ is a solution of

$$
\begin{cases} h_t - b_{ij}D_{ij}h &= g \quad \text{in} \quad Q_4^+, \\ \qquad\qquad h &= 0 \quad \text{on} \quad T_4 \end{cases} \tag{3.1.31}
$$

satisfying

$$
\fint_{Q_4^+} |h_t|^q + |D^2h|^q dz \le 1 \quad \text{and} \quad \fint_{Q_4^+} |g|^q dz \le \delta,
$$

then there exist a constant matrix $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$ with $\|\overline{\mathbf{B}}_{Q_4^+} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \le \epsilon$

and a solution $v \in W_q^{2,1}(Q_4^+)$ *of*

$$\left\{ \begin{array}{rcll} v_t - \tilde{b}_{ij} D_{ij} v & = & 0 & \textit{in} \quad Q_4^+, \\ v & = & 0 & \textit{on} \quad T_4 \end{array} \right. \tag{3.1.32}$$

satisfying

$$⨍_{Q_4^+} |v_t|^q + |D^2 v|^q dz \le 1 \tag{3.1.33}$$

and

$$\int_{Q_4^+} |h - (\overline{D_n h})_{Q_4^+} x_n - v|^q dz \le \epsilon.$$

Proof. We argue by contradiction. If not, there exist $\epsilon_0 > 0$, $h_l \in W_q^{2,1}(Q_4^+)$, $g_l \in L^q(Q_4^+)$ and $\mathbf{B}_l = (b_{ij}^l) : \mathbb{R}^{n+1} \to \mathbb{R}^{n \times n}$, where $l = 1, 2, \ldots,$ such that $\mathbf{B}_l$ is uniformly parabolic with the parabolicity constant Λ satisfying $[\mathbf{B}_l]_4 \le \frac{1}{l}$, and $h_l \in W_q^{2,1}(Q_4^+)$ is a solution of

$$\left\{ \begin{array}{rcll} (h_l)_t - b_{ij}^l D_{ij} h_l & = & g_l & \text{in} \quad Q_4^+, \\ h_l & = & 0 & \text{on} \quad T_4 \end{array} \right.$$

satisfying

$$⨍_{Q_4^+} |(h_l)_t|^q + |D^2 h_l|^q dz \le 1 \quad \text{and} \quad ⨍_{Q_4^+} |g_l|^q dz \le \frac{1}{l}, \tag{3.1.34}$$

but

$$\int_{Q_4^+} |h_l - (\overline{D_n h_l})_{Q_4^+} x_n - v|^q dz > \epsilon_0, \tag{3.1.35}$$

for any constant matrix $\tilde{\mathbf{B}}$ with $\|\overline{\mathbf{B}}_{Q_4^+} - \tilde{\mathbf{B}}\|_{L^\infty(\mathbb{R}^{n+1})} \le \epsilon_0$ and any solution $v \in W_q^{2,1}(Q_4^+)$ of (3.1.32) satisfying (3.1.33).

From the condition $[\mathbf{B}_l]_4 \le \frac{1}{l}$, a simple computation gives

$$\begin{aligned} & ⨍_{Q_4^+} |\mathbf{B}_l - \overline{\mathbf{B}}_{l Q_4^+}| dz \\ & \le 2 ⨍_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{l Q_4}| dz + |\overline{\mathbf{B}}_{l Q_4^+} - \overline{\mathbf{B}}_{l Q_4}| \\ & \le 2 ⨍_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{l Q_4}| dz + 2 ⨍_{Q_4} |\mathbf{B}_l - \overline{\mathbf{B}}_{l Q_4}| dz \le \frac{4}{l}. \end{aligned} \tag{3.1.36}$$

By the same argument as in (3.1.22) along with (3.1.36), we deduce that

$$\mathbf{B}_l \longrightarrow \mathbf{B}_0 \ \text{ in } \ L^{q'}(Q_4^+) \ \text{ as } \ l \to \infty \ \text{ (up to subsequence)}, \tag{3.1.37}$$

for some constant matrix $\mathbf{B}_0 = (b_{ij}^0)$.

We now set $v_l = h_l - (\overline{D_n h_l})_{Q_4^+} x_n$. It is clear that $v_l \in W_q^{2,1}(Q_4^+)$ with $v_l = 0$ on T_4, and then Lemma 3.1.3 implies

$$\begin{aligned} \fint_{Q_4^+} |v_l|^q dz &\le c \fint_{Q_4^+} |(v_l)_t|^q + |D^2 v_l|^q dz \\ &= c \fint_{Q_4^+} |(h_l)_t|^q + |D^2 h_l|^q dz \ \le \ c, \end{aligned} \tag{3.1.38}$$

where the last inequality comes from (3.1.34). In an analogous way to (3.1.24) with (3.1.38) in place of (3.1.23), we also have

$$\int_{Q_4^+} |Dv_l|^q dz \le c \left(\int_{Q_4^+} |v_l|^q dz + \int_{Q_4^+} |D^2 v_l|^q dz \right) \le c. \tag{3.1.39}$$

In turn, it follows from (3.1.34), (3.1.38) and (3.1.39) that $\{v_l\}_{l=1}^\infty$ is bounded in $W_q^{2,1}(Q_4^+)$. Then there exist a subsequence of $\{v_l\}_{l=1}^\infty$, which is still denoted by $\{v_l\}_{l=1}^\infty$, and a function $v_0 \in W_q^{2,1}(Q_4^+)$ such that

$$\begin{cases} v_l \rightharpoonup v_0 & \text{weakly in } W_q^{2,1}(Q_4^+), \\ v_l \to v_0 & \text{strongly in } L^q(Q_4^+), \end{cases} \quad \text{as} \ \ l \to \infty. \tag{3.1.40}$$

By (3.1.34), (3.1.37) and (3.1.40), it is easy to check that $v_0 \in W_q^{2,1}(Q_4^+)$ is a solution of

$$\begin{cases} (v_0)_t - b_{ij}^0 D_{ij} v_0 = 0 & \text{in} \quad Q_4^+, \\ \qquad\qquad\quad v_0 = 0 & \text{on} \quad T_4, \end{cases}$$

satisfying

$$\fint_{Q_4^+} |(v_0)_t|^q + |D^2 v_0|^q dz \le \liminf_{l\to\infty} \fint_{Q_4^+} |(v_l)_t|^q + |D^2 v_l|^q dz \le 1. \tag{3.1.41}$$

However, this is a contradiction to (3.1.35). This completes the proof. $\square$

Corollary 3.1.7. *Under the hypotheses and conclusion of Lemma 3.1.6, we have*

$$\fint_{Q_1^+} |(h - v)_t|^q + |D^2(h - v)|^q dz \le \epsilon.$$

Proof. We proceed as in Corollary 3.1.5 with Lemma 3.1.6 in place of Lemma 3.1.4. □

3.2 Weighted estimates in Orlicz spaces

3.2.1 Assumptions and main result

Before stating our main result, we recall some properties of Muckenhoupt classes A_q for $1 < q < \infty$, which are the primary focus of this chapter. Given $1 < q < \infty$, a nonnegative function $w = w(x,t) \in L^1_{loc}(\mathbb{R}^{n+1})$ is called a *weight* in *Muckenhoupt class* A_q, or an A_q *weight*, denoted by $w \in A_q$, if quantity

$$[w]_q := \sup_Q \left(\fint_Q w(x,t)dxdt \right) \left(\fint_Q w(x,t)^{\frac{-1}{q-1}} dxdt \right)^{q-1} \tag{3.2.1}$$

is finite, where the supremum is taken over all parabolic cylinders $Q \subset \mathbb{R}^{n+1}$. There is an alternative definition of A_q weights as follows; given $1 < q < \infty$, the weight $w \in A_q$ if and only if

$$\left(\fint_Q f(x,t)dxdt \right)^q \leq \frac{c}{w(Q)} \int_Q f(x,t)^q w(x,t)dxdt \tag{3.2.2}$$

holds for all nonnegative measurable functions f and all parabolic cylinders $Q \subset \mathbb{R}^{n+1}$. Here, the smallest c for which (3.2.2) is valid equals $[w]_q$.

Every A_q weight possesses the doubling property. More precisely, whenever $w \in A_q$, it follows from (3.2.2) with $f = \chi_{Q_1(y,s)}$ and $Q = Q_2(y,s)$ for any $(y,s) \in \mathbb{R}^{n+1}$, that

$$w(Q_2(y,s)) \leq 2^{(n+2)q}[w]_q w(Q_1(y,s)).$$

Thanks to the doubling property of A_q weights, we can replace the family of parabolic cylinders by the family of cubes or other such equivalent families in the definition (3.2.1). Furthermore, we remark that the parabolic cylinders are used in (3.2.1) instead of normal cubes, because such a weight w defined in (3.2.1) is appropriate for our problem related to the parabolic equation. In particular, one of basic tools in our approach is the Vitali covering lemma that uses the parabolic cylinders; see Lemmas 3.2.8 and 3.2.9. The A_q class is stable with respect to translation, dilation and multiplication by a positive scalar and has monotonicity $A_{q_1} \subset A_{q_2}$ for $q_1 \leq q_2$.

A typical example of A_q weights for $1 < q < \infty$ is the function

$$w_\alpha(x,t) = \rho(x,t)^\alpha, \quad (x,t) \in \mathbb{R}^{n+1}$$

for $-(n+2) < \alpha < (n+2)(q-1)$, where $\rho(x,t)$ is a parabolic metric, that is,

$$\rho(x,t) = \left(\frac{|x|^2 + \sqrt{|x|^4 + 4|t|^2}}{2} \right)^{\frac{1}{2}}.$$

We identify weight w with measure

$$w(E) = \int_E w(x,t)dxdt,$$

for measurable sets $E \subset \mathbb{R}^{n+1}$. The following are the crucial fundamental properties of the A_q weights whose proofs can be found in [68, 70].

Lemma 3.2.1. *Let w be an A_q weight for some q with $1 < q < \infty$, and let E be a measurable subset of a parabolic cube $Q \subset \mathbb{R}^{n+1}$. Then there exist two constants β, $\nu > 0$ depending only on n and w such that*

$$[w]_q^{-1} \left(\frac{|E|}{|Q|} \right)^q \leq \frac{w(E)}{w(Q)} \leq \beta \left(\frac{|E|}{|Q|} \right)^\nu.$$

Lemma 3.2.2. *(Self-improved Property) Assume $w \in A_q$ for some q with $1 < q < \infty$. Then there exists a sufficiently small constant $\epsilon_0 > 0$, depending only on n, q and $[w]_q$ such that $w \in A_{q-\epsilon_0}$.*

Orlicz spaces have several properties that should be reviewed before proceeding. A nonnegative real-valued function Φ defined on $[0,\infty)$ is called a *Young function*, if it is increasing, convex, and satisfying

$$\Phi(0) = 0, \Phi(\infty) = \lim_{\rho\to\infty} \Phi(\rho) = +\infty, \text{ and } \lim_{\rho\to 0^+} \frac{\Phi(\rho)}{\rho} = \lim_{\rho\to\infty} \frac{\rho}{\Phi(\rho)} = 0.$$

The convexity of Φ gives us

$$\Phi(\lambda\rho) \leq \lambda\Phi(\rho), \text{ for any } \lambda \in [0,1].$$

Throughout this chapter, the Young function Φ is assumed to satisfy the following $\Delta_2 \cap \nabla_2$-condition, which is unavoidable for the type of regularity we are considering; see [72, 73, 75].

Definition 3.2.3. We say that the Young function Φ satisfies the $\Delta_2 \cap \nabla_2$-condition, denoted by $\Phi \in \Delta_2 \cap \nabla_2$, if
(1) (Δ_2-*condition*) for some number $\mu_1 > 1$, $\Phi(2\rho) \le \mu_1 \Phi(\rho)$ for all $\rho > 0$, and
(2) (∇_2-*condition*) for some number $\mu_2 > 1$, $\Phi(\rho) \le \frac{1}{2\mu_2}\Phi(\mu_2 \rho)$ for all $\rho > 0$.

Note that the Δ_2-condition is equivalent to that for every $\lambda > 1$ there exists a positive constant $\mu = \mu(\lambda)$ such that $\Phi(\lambda\rho) \le \mu\Phi(\rho)$ for all $\rho \ge 0$; see [52] for more details. Indeed, if $\Phi \in \Delta_2$, taking $1 < \lambda \le 2^k$ for some $k \ge 1$, we deduce that

$$\Phi(\lambda\rho) \le \Phi(2^k\rho) \le 2^k\Phi(\rho) = \mu\Phi(\rho)$$

and conversely, letting $\lambda \ge 2^{-k}$ for some $k \ge 1$, one has that

$$\Phi(2\rho) \le \Phi(\lambda^k\rho) \le \mu^k\Phi(\rho),$$

which implies $\Phi \in \Delta_2$.

Next, consider the function

$$h_\Phi(\lambda) = \sup_{\rho>0} \frac{\Phi(\lambda\rho)}{\Phi(\rho)} \quad \text{for } \lambda > 0,$$

and define the *lower index* of Φ by

$$i(\Phi) = \lim_{\lambda\to 0^+} \frac{\log(h_\Phi(\lambda))}{\log\lambda} = \sup_{0<\lambda<1} \frac{\log(h_\Phi(\lambda))}{\log\lambda}.$$

We observe that since $\Phi \in \Delta_2$, there exist two constants q_1 and q_2 with $1 < q_1 \le q_2 < \infty$ such that

$$c^{-1}\min\{\lambda^{q_1}, \lambda^{q_2}\}\Phi(\rho) \le \Phi(\lambda\rho) \le c\max\{\lambda^{q_1}, \lambda^{q_2}\}\Phi(\rho), \tag{3.2.3}$$

for $\lambda, \rho \ge 0$, where the constant c is independent of λ and ρ (see [47]). It is worth noticing that $i(\Phi)$ is equal to the supremum of those q_1 satisfying the above inequality (3.2.3) with $\lambda \ge 1$. A simple example of the Young function Φ satisfying the $\Delta_2 \cap \nabla_2$-condition is $\Phi(\rho) = \rho^q$ with $q > 1$, and in this case, we see that $i(\Phi) = q$.

Related to Young function $\Phi \in \Delta_2 \cap \nabla_2$ and weight $w = w(x,t)$ is the *weighted Orlicz space* $L^\Phi_w(\Omega_T)$, which consists of all Lebesgue measurable functions g on Ω_T such that

$$\int_{\Omega_T} \Phi(|g(x,t)|)w(x,t)dxdt < +\infty.$$

In particular, this *weighted Orlicz space* $L^\Phi_w(\Omega_T)$ is a weighted rearranged invariant Banach space equipped with the following *Luxemburg norm*

$$\|g\|_{L^\Phi_w(\Omega_T)} = \inf\left\{\lambda > 0 : \int_{\Omega_T} \Phi\left(\frac{|g(x,t)|}{\lambda}\right) w(x,t)dxdt \leq 1\right\}.$$

Moreover, we define the *weighted Orlicz Sobolev space* $W^{2,1}L^\Phi_w(\Omega_T)$ as the set of functions $u(x,t)$ in $L^\Phi_w(\Omega_T)$ whose distributional derivatives $D^r_t D^s_x u(x,t)$ with $0 \leq 2r+s \leq 2$ also belong to $L^\Phi_w(\Omega_T)$, and its norm is naturally given by

$$\|u\|_{W^{2,1}L^\Phi_w(\Omega_T)} = \sum_{j=0}^{2} \sum_{2r+s=j} \|D^r_t D^s_x u\|_{L^\Phi_w(\Omega_T)}.$$

We refer to [47, 51, 43] for more details on weighted Orlicz spaces.

For the special case in which $w(x,t) \equiv 1$, and either $\Phi(\rho) = \rho^p$ if $1 \leq p < \infty$, or $\Phi(\rho) = 0$ for $0 \leq \rho \leq 1$ and $\Phi(\rho) = \infty$ for $\rho > 1$ if $p = \infty$, the spaces $L^\Phi_w(\Omega_T)$ and $W^{2,1}L^\Phi_w(\Omega_T)$ coincide with the classical Lebesgue and Sobolev spaces $L^p(\Omega_T)$ and $W^{2,1}_p(\Omega_T)$, respectively. Here, the Sobolev space $W^{2,1}_p(\Omega_T)$ consists of functions $u \in L^p(\Omega_T)$ such that $u_t, Du, D^2u \in L^p(\Omega_T)$. Moreover, considering the case in which $w(x,t) \equiv 1$, we also see that the Orlicz spaces $L^\Phi(\Omega_T)$ and $W^{2,1}L^\Phi(\Omega_T)$ are special cases of the spaces $L^\Phi_w(\Omega_T)$ and $W^{2,1}L^\Phi_w(\Omega_T)$, respectively.

From the classical Calderón-Zygmund theory, it is well known that the second order derivatives of solutions u to the Poisson equation $\Delta u = f \in L^1(\Omega)$ (respectively, L^∞) do not belong to the space $L^1(\Omega)$ (respectively, $L^\infty(\Omega)$) in general. We point out that the L^1 space is close to the Orlicz space not satisfying the ∇_2 condition and the L^∞ space is close to the Orlicz space not satisfying the Δ_2 condition.

However, this Calderón-Zygmund theory is still valid in the setting of Orlicz spaces satisfying both Δ_2 and ∇_2 conditions, i.e., the second order derivatives of solutions u to $\Delta u = f \in L^\Phi(\Omega)$ lie in the Orlicz space $L^\Phi(\Omega)$, under the assumption that Φ satisfies the $\Delta_2 \cap \nabla_2$-condition. This observation motivated our work to consider the Orlicz spaces. Indeed, this type of explicit information inspired us to replace the Lebesgue spaces by the Orlicz spaces. On the other hand, there have been several studies on the regularity theory for solutions to elliptic and parabolic equations in the setting of Orlicz spaces as natural generalizations of the classical Lebesgue and Sobolev spaces; see, for instance, [17, 72, 73].

Given weight w and Young function Φ, our main assumption is

$$w = w(x,t) \in A_{i(\Phi)} \quad \text{for } \Phi \in \Delta_2 \cap \nabla_2. \tag{3.2.4}$$

This assumption (3.2.4) is the necessary and sufficient condition under which the Hardy-Littlewood maximal operator is bounded on the corresponding weighted Orlicz space; see [47, Theorem 2.1.1]. Furthermore, since $\Phi \in \nabla_2$ implies $i(\Phi) > 1$ (see [37]), the previously mentioned properties of the Muckenhoupt weight can be applied to the proof of our main result under assumption (3.2.4). From the definition of the Luxemburg norm and inequality (3.2.3), one can readily check that

$$\frac{1}{c}\left(\|g\|^{\alpha}_{L^{\Phi}_{w}(\Omega_T)} - 1\right) \leq \int_{\Omega_T} \Phi\left(|g(x,t)|\right) w(x,t) dxdt \leq c\left(\|g\|^{\beta}_{L^{\Phi}_{w}(\Omega_T)} + 1\right), \tag{3.2.5}$$

where $\alpha = q_1$ and $\beta = q_2$ satisfy (3.2.3) and the constant $c > 1$ is independent of g.

Whenever f belongs to a suitable space $L^p(\Omega_T)$ with $p \geq 1$, we say that u is a *strong solution* of the equation in (3.0.1) if $u \in W^{2,1}_p(\Omega_T)$, the equation is satisfied almost everywhere in Ω_T, and the boundary condition holds in the sense of trace on $\partial_p \Omega_T$. Throughout the thesis, we shall consider the strong solutions of the parabolic equations.

Let us now state one of the main theorems in this chapter.

Theorem 3.2.4 (Main Theorem)**.** *Given any Young function $\Phi \in \Delta_2 \cap \nabla_2$, let $w = w(x,t) \in A_{i(\Phi)}$. Then there exists a small $\delta = \delta(\Lambda, n, \Phi, w, \partial\Omega) > 0$ so that if $\mathbf{A}$ is uniformly parabolic and (δ, R)-vanishing, $\partial\Omega \in C^{1,1}$ and $|f|^2 \in L^{\Phi}_w(\Omega_T)$, then the problem* (3.0.1) *has a unique strong solution u which satisfies $|u|^2, |u_t|^2, |Du|^2$, $|D^2 u|^2 \in L^{\Phi}_w(\Omega_T)$ with estimate*

$$\begin{aligned} &\left\||u|^2\right\|_{L^{\Phi}_w(\Omega_T)} + \left\||u_t|^2\right\|_{L^{\Phi}_w(\Omega_T)} + \left\||Du|^2\right\|_{L^{\Phi}_w(\Omega_T)} + \left\||D^2u|^2\right\|_{L^{\Phi}_w(\Omega_T)} \\ &\leq c\left\||f|^2\right\|_{L^{\Phi}_w(\Omega_T)} \end{aligned}$$

for some constant $c > 0$ being independent of u and f.

Remark 3.2.5. From $|f|^2 \in L^{\Phi}_w(\Omega_T) \subset L^1(\Omega_T)$ (see (3.2.22)), Theorem 4.3 in [9] ensures the existence of a unique strong solution $u \in W^{2,1}_2(\Omega_T)$ to (3.0.1).

Since the Young function $\Phi(\rho) = \rho^{\frac{p}{2}}$ with $p > 2$ has the $\Delta_2 \cap \nabla_2$-condition and weight $w(x,t) \equiv 1$ obviously belongs to the $A_{i(\Phi)}$ class, we note that the L^p regularity in [9] is a special case of Theorem 3.2.4 when $\Phi(\rho) = \rho^{\frac{p}{2}}$ with $p > 2$ and $w(x,t) \equiv 1$.

3.2.2 Preliminaries

In this section, we deal primarily with the Hardy-Littlewood maximal function and the Vitali covering lemma, which play key roles in our approach. We first recall that for a locally integrable function $g : \mathbb{R}^{n+1} \to \mathbb{R}$, the Hardy-Littlewood maximal function of g is defined by

$$\mathcal{M}g(y,s) = \sup_{r>0} \frac{1}{|Q_r(y,s)|} \int_{Q_r(y,s)} |g(x,t)| dxdt,$$

at each point $(y,s) \in \mathbb{R}^{n+1}$, where $Q_r(y,s)$ is the parabolic cylinder. If the definition of g is restricted to a bounded domain $\mathcal{D} \subset \mathbb{R}^{n+1}$, we use

$$\mathcal{M}_{\mathcal{D}} g = \mathcal{M}(\chi_{\mathcal{D}} g),$$

where $\chi_{\mathcal{D}}$ is the characteristic function of $\mathcal{D}$. The maximal function $\mathcal{M}$ is of weak type $(1,1)$ and of strong type (p,p) for $1 < p \leq \infty$, that is,

$$\left|\left\{(x,t) \in \mathbb{R}^{n+1} : \mathcal{M}g(x,t) \geq \lambda\right\}\right| \leq \frac{c_1}{\lambda} \|g\|_{L^1(\mathbb{R}^{n+1})} \quad \text{for} \quad \forall \lambda > 0, \tag{3.2.6}$$

and

$$\|\mathcal{M}g\|_{L^p(\mathbb{R}^{n+1})} \leq c_2 \|g\|_{L^p(\mathbb{R}^{n+1})} \quad \text{for} \quad 1 < p \leq \infty, \tag{3.2.7}$$

where the constant c_1 depends only on n and the constant c_2 depends only on n and p. It is well known that the Hardy-Littlewood maximal function is bounded from the weighted L^p space L^p_w to itself, with $1 < p < \infty$, if and only if $w \in A_p$; see [60, 68, 70]. This boundedness in L^p_w has been extended to weighted Orlicz spaces by Kerman and Torchinsky in [44] as follows.

Lemma 3.2.6. *Suppose that Φ is a Young function satisfying the $\Delta_2 \cap \nabla_2$-condition and that $w = w(x,t) \in A_{i(\Phi)}$. Then there exists a positive constant $c = c(n, \Phi, w)$ such that*

$$\int_{\mathbb{R}^{n+1}} \Phi(|g|) w \; dxdt \leq \int_{\mathbb{R}^{n+1}} \Phi(\mathcal{M}g) w \; dxdt \leq c \int_{\mathbb{R}^{n+1}} \Phi(|g|) w \; dxdt$$

for all $g = g(x,t) \in L^\Phi_w(\mathbb{R}^{n+1})$.

We also use the following standard property, which comes from classical measure theory of weighted Orlicz spaces; see [17, 22, 47, 63].

Lemma 3.2.7. *Assume g is a nonnegative measurable function in Ω_T, and let $\eta > 0$ and $M > 1$ be constants. For any $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_q$ with some $1 < q < \infty$, we have that $g \in L^\Phi_w(\Omega_T)$ if and only if*

$$S := \sum_{k \geq 1} \Phi(M^k) w \left(\left\{ (x,t) \in \Omega_T : g(x,t) > \eta M^k \right\} \right) < \infty,$$

and moreover,

$$c^{-1} S \leq \int_{\Omega_T} \Phi\left(|g(x,t)|\right) w(x,t) dxdt \leq c \left(w(\Omega_T) + S \right),$$

where the constant $c > 0$ depends only on η, M, Φ, q and $[w]_q$.

We next consider the following modified versions of the Vitali covering lemma, which will be applied later in this chapter to the proofs of the interior and boundary weighted Orlicz estimates.

Lemma 3.2.8. *Let w be an A_q weight for some q with $1 < q < \infty$. Let $0 < \epsilon < 1$ and suppose that the measurable sets E and F with $E \subset F \subset Q_1$ satisfy the following properties:*
(1) $w(E) < \epsilon w(Q_1)$, and
(2) for every $(x,t) \in Q_1$ and $0 < r \leq 1$,

$$w\left(E \cap Q_r(x,t)\right) \geq \epsilon w\left(Q_r(x,t)\right) \text{ implies } Q_r(x,t) \cap Q_1 \subset F.$$

Then $w(E) \leq 10^{(n+2)q} \epsilon [w]_q^2 w(F)$.

Proof. In view of (1), we know that for almost all $(x,t) \in E$, there is a small $\rho_{(x,t)} > 0$ such that

$$w(E \cap Q_{\rho_{(x,t)}}(x,t)) = \epsilon w(Q_{\rho_{(x,t)}}(x,t)) \text{ and } w(E \cap Q_\rho(x,t)) < \epsilon w(Q_\rho(x,t)) \tag{3.2.8}$$

for all $\rho \in (\rho_{(x,t)}, 1]$. Since $\{Q_{\rho_{(x,t)}}(x,t)\}_{(x,t)\in E}$ covers E, the Vitali covering lemma gives that there is a countable $\{(x_i,t_i)\}_{i=1}^\infty$ so that parabolic cubes $Q_{\rho_{(x_i,t_i)}}(x_i,t_i)$ are mutually disjoint and $E \subset \cup_i Q_{5\rho_{(x_i,t_i)}}(x_i,t_i)$. Then from Lemma 3.2.1 and (3.2.8), we have

$$w(E \cap Q_{5\rho_{(x_i,t_i)}}(x_i,t_i)) < \epsilon w(Q_{5\rho_{(x_i,t_i)}}(x_i,t_i)) \leq \epsilon [w]_q 5^{(n+2)q} w(Q_{\rho_{(x_i,t_i)}}(x_i,t_i)).$$

Note that

$$\sup_{0<\rho\leq 1}\sup_{(x,t)\in Q_1}\frac{|Q_\rho(x,t)|}{|Q_\rho(x,t)\cap Q_1|}\leq 2^{n+2}. \tag{3.2.9}$$

Thus, it follows from Lemma 3.2.1 that

$$\begin{aligned}
w(E) &\leq w\big(E\cap\bigcup_{i\geq 1}Q_{5\rho_{(x_i,t_i)}}(x_i,t_i)\big)\leq\sum_{i\geq 1}\epsilon[w]_s 5^{(n+2)s}w(Q_{\rho_{(x_i,t_i)}}(x_i,t_i))\\
&\leq [w]_q^2 5^{(n+2)q}\epsilon\sum_{i\geq 1}\left(\frac{|Q_{\rho_{(x_i,t_i)}}(x_i,t_i)|}{|Q_{\rho_{(x_i,t_i)}}(x_i,t_i)\cap Q_1|}\right)^q w(Q_{\rho_{(x_i,t_i)}}(x_i,t_i)\cap Q_1)\\
&\leq [w]_q^2 5^{(n+2)q}2^{(n+2)q}\epsilon\sum_{i\geq 1}w(Q_{\rho_{(x_i,t_i)}}(x_i,t_i))\cap Q_1)\\
&\leq [w]_q^2 10^{(n+2)q}\epsilon w\left(\bigcup_{i\geq 1}Q_{\rho_{(x_i,t_i)}}(x_i,t_i)\cap Q_1\right)\\
&\leq 10^{(n+2)q}\epsilon[w]_q^2 w(F),
\end{aligned}$$

where the last inequality comes from (3.2.8) and hypothesis *(2)*. □

Lemma 3.2.9. *Let w be an A_q weight for some q with $1<q<\infty$. Let $0<\epsilon<1$ and suppose that the measurable sets E and F with $E\subset F\subset Q_1^+$ satisfy the following properties:*
(1) $w(E)<\epsilon w(Q_1^+)$, and
(2) for every $(x,t)\in Q_1^+$ and $0<r\leq 1$,

$$w(E\cap Q_r(x,t))\geq\epsilon w(Q_r(x,t)) \text{ implies } Q_r(x,t)\cap Q_1^+\subset F.$$

Then $w(E)\leq 2^q 10^{(n+2)q}\epsilon[w]_q^2 w(F)$.

Proof. This can be proved in exactly the same way as in the proof of Lemma 3.2.8 except that it uses the fact

$$\sup_{0<\rho\leq 1}\sup_{(x,t)\in Q_1^+}\frac{|Q_\rho(x,t)|}{|Q_\rho(x,t)\cap Q_1^+|}\leq 2^{n+3}$$

instead of (3.2.9). □

3.2.3 Interior and boundary weighted Orlicz estimates

In this section, we derive the interior and boundary weighted Orlicz estimates for nondivergence type parabolic equation (3.0.1).

We start with the interior and boundary $W^{2,1}_2$ estimates for equation (3.0.1) found in [9].

Lemma 3.2.10. *There exists a small $\delta = \delta(\Lambda, n) > 0$ such that the following hold:*

(i) (Interior estimates) If $\mathbf{A}$ is uniformly parabolic and $(\delta, 6)$-vanishing and if $f \in L^2(Q_6)$, then for any solution $u \in W^{2,1}_2(Q_6)$ of

$$u_t - a_{ij}D_{ij}u = f \quad \text{in } Q_6,$$

we have

$$\|u_t\|_{L^2(Q_1)} + \|D^2u\|_{L^2(Q_1)} \le c\left(\|f\|_{L^2(Q_6)} + \|u\|_{L^2(Q_6)}\right)$$

for some constant $c > 0$ being independent of u and f.

(ii) (Boundary estimates) If $\mathbf{A}$ is uniformly parabolic and $(\delta, 6)$-vanishing and if $f \in L^2(Q_6^+)$, then for any solution $u \in W^{2,1}_2(Q_6^+)$ of

$$\begin{cases} u_t - a_{ij}D_{ij}u &= f \quad \text{in} \quad Q_6^+, \\ u &= 0 \quad \text{on} \quad T_6, \end{cases}$$

we have

$$\|u_t\|_{L^2(Q_1^+)} + \|D^2u\|_{L^2(Q_1^+)} \le c\left(\|f\|_{L^2(Q_6^+)} + \|u\|_{L^2(Q_6^+)}\right)$$

for some constant $c > 0$ being independent of u and f.

The main theorem in this section follows.

Theorem 3.2.11. *Given any Young function $\Phi \in \Delta_2 \cap \nabla_2$, let $w = w(x,t) \in A_{i(\Phi)}$ and $\rho > 0$. Then there exists a small $\delta = \delta(\Lambda, n, \Phi, w) > 0$ such that the following hold:*

(i) (Interior weighted Orlicz estimates) If $\mathbf{A}$ is uniformly parabolic and $(\delta, 6\rho)$-vanishing and if $|f|^2 \in L^\Phi_w(Q_{6\rho})$, then for any solution $u \in W^{2,1}_2(Q_{6\rho})$ of

$$u_t - a_{ij}D_{ij}u = f \quad \text{in } Q_{6\rho},$$

we have $|u_t|^2$, $|D^2u|^2 \in L^\Phi_w(Q_\rho)$ with estimate

$$\left\||u_t|^2\right\|_{L^\Phi_w(Q_\rho)} + \left\||D^2u|^2\right\|_{L^\Phi_w(Q_\rho)} \le c\left(\left\||f|^2\right\|_{L^\Phi_w(Q_{6\rho})} + \frac{1}{\rho^4}\|u\|^2_{L^2(Q_{6\rho})}\right), \tag{3.2.10}$$

for some constant $c > 0$ being independent of u and f.

(ii) *(Boundary weighted Orlicz estimates) If* $\mathbf{A}$ *is uniformly parabolic and $(\delta, 6\rho)$-vanishing and if $|f|^2 \in L^{\Phi}_{w}(\Omega_T)$, then for any solution $u \in W^{2,1}_2(\Omega_T)$ of*

$$\begin{cases} u_t - a_{ij}D_{ij}u &= f \quad in \quad \Omega_T \supset Q^+_{6\rho}, \\ \qquad\quad u &= 0 \quad on \quad \partial_p\Omega_T \supset T_{6\rho}, \end{cases} \tag{3.2.11}$$

we have $|u_t|^2$, $|D^2u|^2 \in L^{\Phi}_{w}(Q^+_{\rho})$ with estimate

$$\left\||u_t|^2\right\|_{L^{\Phi}_{w}(Q^+_{\rho})} + \left\||D^2u|^2\right\|_{L^{\Phi}_{w}(Q^+_{\rho})} \leq c\left(\left\||f|^2\right\|_{L^{\Phi}_{w}(Q^+_{6\rho})} + \frac{1}{\rho^4}\|u\|^2_{L^2(Q^+_{6\rho})}\right), \tag{3.2.12}$$

for some constant $c > 0$ being independent of u and f.

The interior weighted Orlicz estimates (3.2.10) can be obtained by a similar way as in the proof of the boundary weighted Orlicz estimates (3.2.12), applying Lemma 3.1.4 and Corollary 3.1.5 in place of Lemma 3.1.6 and Corollary 3.1.7, respectively. Hence, we only derive the boundary estimates (3.2.12) in the main theorem, Theorem 3.2.11.

In order to prove the estimates (3.2.12), we also need a series of lemmas as follows.

Lemma 3.2.12. *There is a positive constant $N_1 = N_1(n, \Lambda)$ so that for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, \Lambda, n) > 0$ such that if $u \in W^{2,1}_2(\Omega_T)$ is a solution of*

$$\begin{cases} u_t - a_{ij}D_{ij}u &= f \quad in \quad \Omega_T \supset Q^+_{6}, \\ \qquad\quad u &= 0 \quad on \quad \partial_p\Omega_T \supset T_{6}, \end{cases} \tag{3.2.13}$$

with

$$\begin{aligned} Q^+_1 \cap \{(x,t) \in \Omega_T : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) \leq 1\} \\ \cap\{(x,t) \in \Omega_T : \mathcal{M}(|f|^2)(x,t) \leq \delta^2\} \neq \emptyset, \end{aligned} \tag{3.2.14}$$

and if $\mathbf{A}$ *is uniformly parabolic and $(\delta, 6)$-vanishing, then we have*

$$|\{(x,t) \in \Omega_T : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\} \cap Q^+_1| < \epsilon|Q^+_1|.$$

Proof. From condition (3.2.14), there exists a point $(x_0, t_0) \in Q^+_1$ so that

$$\frac{1}{|Q_\rho|}\int_{Q^+_\rho(x_0,t_0)\cap\Omega_T} |u_t|^2 + |D^2u|^2 dxdt \leq 1$$

$$\text{and} \quad \frac{1}{|Q_\rho|}\int_{Q_\rho^+(x_0,t_0)\cap\Omega_T}|f|^2dxdt \le \delta^2 \quad \text{for all } \rho>0.$$

Note that $Q_4^+ \subset Q_5^+(x_0,t_0)$ and then we see

$$\begin{aligned}&\fint_{Q_4^+}|u_t|^2+|D^2u|^2dxdt\\ &\quad\le \left(\frac{5}{4}\right)^{n+2}\fint_{Q_5^+(x_0,t_0)}|u_t|^2+|D^2u|^2dxdt \le 2^{n+2}.\end{aligned}$$

Similarly, we get

$$\fint_{Q_4^+}|f|^2dxdt \le 2^{n+2}\delta^2.$$

Apply Corollary 3.1.7 to equation (3.2.13) with u and f replaced by $\left(\frac{1}{2}\right)^{\frac{n+2}{2}}u$ and $\left(\frac{1}{2}\right)^{\frac{n+2}{2}}f$ respectively, in order to find that for any $\eta>0$, there exist a small $\delta=\delta(\eta)>0$, a positive constant $N_0=N_0(n,\Lambda)$, a constant matrix $\tilde{\mathbf{A}}=(\tilde{a}_{ij})$ satisfying $\|\tilde{\mathbf{A}}-\overline{\mathbf{A}}_{Q_4^+}\|_{L^\infty}\le\eta$ and a solution $v\in W_2^{2,1}(Q_{\frac{7}{2}}^+)$ of

$$\begin{cases} v_t-\tilde{a}_{ij}D_{ij}v = 0 & \text{in} \quad Q_{\frac{7}{2}}^+,\\ \qquad\qquad v = 0 & \text{on} \quad T_{\frac{7}{2}}\end{cases}$$

such that

$$\|v_t\|^2_{L^\infty(Q_{\frac{7}{2}}^+)}+\|D^2v\|^2_{L^\infty(Q_{\frac{7}{2}}^+)}\le N_0^2$$

and

$$\fint_{Q_3^+}|(u-v)_t|^2+|D^2(u-v)|^2dxdt\le\eta^2,$$

provided

$$\fint_{Q_4^+}|f|^2+|\mathbf{A}-\overline{\mathbf{A}}_{Q_4^+}|^2dxdt\le\delta^2.$$

Next, we claim that

$$\begin{aligned}&\{(x,t)\in Q_1^+:\mathcal{M}(|u_t|^2+|D^2u|^2)(x,t)>N_1^2\} \qquad (3.2.15)\\ &\quad\subset\{(x,t)\in Q_1^+:\mathcal{M}_{Q_3^+}(|(u-v)_t|^2+|D^2(u-v)|^2)(x,t)>N_0^2\},\end{aligned}$$

where $N_1:=\max\{4N_0^2,2^{n+2}\}$. In order to verify this, let us suppose

$$(x_1,t_1)\in\{(x,t)\in Q_1^+:\mathcal{M}_{Q_3^+}(|(u-v)_t|^2+|D^2(u-v)|^2)(x,t)\le N_0^2\}.$$

Then for $\rho \leq 2$, we see $Q^+_\rho(x_1,t_1) \subset Q^+_3$ and so

$$\begin{aligned}
&\frac{1}{|Q_\rho|}\int_{Q^+_\rho(x_1,t_1)} |u_t|^2 + |D^2u|^2 dxdt \\
&\leq \frac{1}{|Q_\rho|}\int_{Q^+_\rho(x_1,t_1)} 2\left(|(u-v)_t|^2 + |D^2(u-v)|^2\right) + 2(|v_t|^2 + |D^2v|^2) dxdt \\
&\leq 2\mathcal{M}_{Q^+_3}\left(|(u-v)_t|^2 + |D^2(u-v)|^2\right)(x_1,t_1) + 2N_0^2 \leq 4N_0^2.
\end{aligned}$$

On the other hand, if $\rho > 2$, $(x_0,t_0) \in Q^+_\rho(x_1,t_1) \subset Q^+_{2\rho}(x_0,t_0)$, and then we get

$$\begin{aligned}
&\frac{1}{|Q_\rho|}\int_{Q^+_\rho(x_1,t_1)\cap\Omega_T} |u_t|^2 + |D^2u|^2 dxdt \\
&\quad \leq \frac{2^{n+2}}{|Q_{2\rho}|}\int_{Q^+_{2\rho}(x_0,t_0)\cap\Omega_T} |u_t|^2 + |D^2u|^2 dxdt \leq 2^{n+2}.
\end{aligned}$$

Therefore we can infer

$$(x_1,t_1) \in \left\{(x,t) \in Q^+_1 : \mathcal{M}\left(|u_t|^2 + |D^2u|^2\right)(x,t) \leq N_1^2\right\},$$

and claim (3.2.15) is proved.

Hence, from (3.2.15) and the weak 1-1 estimate (3.2.6), we finally obtain that for some positive constant $c = c(n,\Lambda)$,

$$\begin{aligned}
&\frac{1}{|Q^+_1|}|\{(x,t) \in Q^+_1 : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\}| \\
&\leq \frac{1}{|Q^+_1|}|\{(x,t) \in Q^+_1 : \mathcal{M}_{Q^+_3}(|(u-v)_t|^2 + |D^2(u-v)|^2)(x,t) > N_0^2\}| \\
&\leq c \fint_{Q^+_3} |(u-v)_t|^2 + |D^2(u-v)|^2 dxdt \leq c\eta^2 < \epsilon,
\end{aligned}$$

if we take η and δ so that the last inequality is satisfied. □

Lemma 3.2.13. *Let w be an A_q weight in $\mathbb{R}^{n+1}$ for some q with $1 < q < \infty$. Then there is a positive constant $N_1 = N_1(\Lambda, n)$ so that for any $\epsilon > 0$ and for every $0 < r \leq 1$, there exists a small $\delta = \delta(\epsilon, \Lambda, n, w, q) > 0$ such that if $u \in W^{2,1}_2(\Omega_T)$ is a solution of*

$$\left\{\begin{array}{rcll} u_t - a_{ij}D_{ij}u &=& f & \text{in} \quad \Omega_T \supset Q^+_{6r}, \\ u &=& 0 & \text{on} \quad \partial_p\Omega_T \supset T_{6r}, \end{array}\right.$$

with

$$Q_r^+ \cap \{(x,t) \in \Omega_T : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) \leq 1\} \\ \cap\{(x,t) \in \Omega_T : \mathcal{M}(|f|^2)(x,t) \leq \delta^2\} \neq \emptyset, \qquad (3.2.16)$$

and if $\mathbf{A}$ *is uniformly parabolic and* $(\delta, 6r)$*-vanishing, then*

$$w\left(\{(x,t) \in \Omega_T : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\} \cap Q_r^+\right) < \epsilon w(Q_r^+).$$

Proof. First, we define $\tilde{a}_{ij}(x,t) = a_{ij}(rx, r^2t)$, $\tilde{u}(x,t) = \frac{1}{r^2}u(rx, r^2t)$, $\tilde{f}(x,t) = f(rx, r^2t)$ and $\tilde{\Omega}_T = \{\left(\frac{1}{r}x, \frac{1}{r^2}t\right) : (x,t) \in \Omega_T\}$. Then we observe that $\tilde{u} \in W_2^{2,1}(\tilde{\Omega}_T)$ is a solution of

$$\begin{cases} \tilde{u}_t - \tilde{a}_{ij}D_{ij}\tilde{u} = \tilde{f} & \text{in} \quad \tilde{\Omega}_T \supset Q_6^+, \\ \tilde{u} = 0 & \text{on} \quad \partial_p\tilde{\Omega}_T \supset T_6. \end{cases}$$

Let $\epsilon > 0$ be given and choose $\delta = \delta(\epsilon, \Lambda, n, w, q)$ as in Lemma 3.2.12 with ϵ replaced by $(\frac{\epsilon}{2\beta})^{\frac{1}{\nu}}$, where β and ν are the constants found in Lemma 3.2.1. On the other hand, hypothesis (3.2.16) asserts that there exists a point

$$(x_0,t_0) \in Q_r^+ \cap \{(x,t) \in \Omega_T : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) \leq 1\} \\ \cap\{(x,t) \in \Omega_T : \mathcal{M}(|f|^2)(x,t) \leq \delta^2\}.$$

Then we see that

$$z_0 := \left(\frac{1}{r}x_0, \frac{1}{r^2}t_0\right) \in Q_1^+ \cap \{z \in \tilde{\Omega}_T : \mathcal{M}(|\tilde{u}_t|^2 + |D^2\tilde{u}|^2)(z) \leq 1\} \\ \cap\{z \in \tilde{\Omega}_T : \mathcal{M}(|\tilde{f}|^2)(z) \leq \delta^2\}.$$

Note that all the hypotheses of Lemma 3.2.12 are fulfilled, which provides the following:

$$|\{z \in \tilde{\Omega}_T : \mathcal{M}(|\tilde{u}_t|^2 + |D^2\tilde{u}|^2)(z) > N_1^2\} \cap Q_1^+| < \left(\frac{\epsilon}{2\beta}\right)^{\frac{1}{\nu}} |Q_1^+|.$$

This implies

$$|\{(x,t) \in \Omega_T : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\} \cap Q_r^+| < \left(\frac{\epsilon}{2\beta}\right)^{\frac{1}{\nu}} |Q_r^+|. \qquad (3.2.17)$$

Hence, from Lemma 3.2.1 and the inequality (3.2.17), we conclude that

$$\begin{aligned}
& w\left(\{(x,t) \in \Omega_T : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\} \cap Q_r^+\right) \\
& \leq \beta \left(\frac{|\{(x,t) \in \Omega_T : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\} \cap Q_r^+|}{|Q_r^+|} \right)^{\nu} w(Q_r^+) \\
& \leq \frac{\epsilon}{2} w(Q_r^+) < \epsilon w(Q_r^+).
\end{aligned}$$

□

Lemma 3.2.14. *Let w be an A_q weight in $\mathbb{R}^{n+1}$ for some q with $1 < q < \infty$. Then there is a constant $N_1 = N_1(\Lambda, n) > 1$ so that for any $\epsilon > 0$, $0 < r \leq \frac{1}{18}$ and $(y,s) \in Q_1^+$, there exists a small $\delta = \delta(\epsilon, \Lambda, n, w, q) > 0$ such that if $u \in W_2^{2,1}(\Omega_T)$ is a solution of*

$$\begin{cases} u_t - a_{ij}D_{ij}u = f & \text{in} \quad \Omega_T \supset Q_6^+, \\ u = 0 & \text{on} \quad \partial_p\Omega_T \supset T_6, \end{cases}$$

with

$$w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\} \cap Q_r(y,s)\right) \geq \epsilon w(Q_r(y,s)) \tag{3.2.18}$$

and if $\mathbf{A}$ is uniformly parabolic and $(\delta, 6)$-vanishing, then

$$\begin{aligned} Q_r(y,s) \cap Q_1^+ \quad \subset \quad & \{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > 1\} \\ & \cup \{(x,t) \in Q_1^+ : \mathcal{M}(|f|^2)(x,t) > \delta^2\}. \end{aligned} \tag{3.2.19}$$

Proof. We argue by contradiction. Let us assume that (3.2.18) holds and (3.2.19) is false. Then there exists a point

$$(x_0, t_0) := (x_0{}', x_{0n}, t_0) \in Q_r(y,s) \cap Q_1^+$$

such that

$$\frac{1}{|Q_\rho|} \int_{Q_\rho^+(x_0,t_0) \cap \Omega_T} |u_t|^2 + |D^2u|^2 dxdt \leq 1 \text{ and } \frac{1}{|Q_\rho|} \int_{Q_\rho^+(x_0,t_0) \cap \Omega_T} |f|^2 dxdt \leq \delta^2$$

for any $\rho > 0$.

We only consider the case in which $Q_{6r}(x_0,t_0) \not\subset Q_6^+$, that is, $Q_{6r}(x_0,t_0) \cap T_6 \neq \emptyset$, because the conclusion when $Q_{6r}(x_0,t_0) \subset Q_6^+$ can be proved in the same way as the case $Q_{6r}(x_0,t_0) \not\subset Q_6^+$.

Then it is easy to see that $(x_0{}', 0, t_0) \in T_1$, and moreover, that

$$Q_r(y,s) \cap Q_1^+ \subset Q_{6r}^+(x_0,t_0) \subset Q_{12r}^+(x_0{}',0,t_0) \subset Q_{72r}^+(x_0{}',0,t_0) \subset Q_6^+ \subset \Omega_T,$$

for $0 < r \leq \frac{1}{18}$. Then we apply Lemma 3.2.13 to $Q_{12r}^+(x_0{}',0,t_0)$ with ϵ replaced by $\frac{2^\nu \epsilon}{\beta[w]_q \left(\frac{3}{5}\right)^{(n+2)\nu} 20^{(n+2)q}}$, in order to attain

$$\begin{aligned}
&\frac{w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\} \cap Q_r(y,s)\right)}{w\left(Q_r(y,s)\right)} \\
&\leq \frac{w\left(\{(x,t) \in Q_{12r}^+(x_0{}',0,t_0) : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\}\right)}{w\left(Q_r(y,s)\right)} \\
&< \frac{2^\nu \epsilon w\left(Q_{12r}^+(x_0{}',0,t_0)\right)}{\beta[w]_q \left(\frac{3}{5}\right)^{(n+2)\nu} 20^{(n+2)q} w\left(Q_r(y,s)\right)}.
\end{aligned} \tag{3.2.20}$$

However, since $Q_{12r}^+(x_0{}',0,t_0) \subset Q_{20r}(y,s)$, Lemma 3.2.1 yields

$$\begin{aligned}
&w\left(Q_{12r}^+(x_0{}',0,t_0)\right) \\
&\leq \beta \left(\frac{|Q_{12r}^+|}{|Q_{20r}|}\right)^\nu w\left(Q_{20r}(y,s)\right) \\
&\leq \beta 2^{-\nu} \left(\frac{3}{5}\right)^{(n+2)\nu} w\left(Q_{20r}(y,s)\right) \\
&\leq \beta 2^{-\nu} \left(\frac{3}{5}\right)^{(n+2)\nu} [w]_q \left(\frac{|Q_{20r}(y,s)|}{|Q_r(y,s)|}\right)^q w(Q_r(y,s)) \\
&= \beta 2^{-\nu} \left(\frac{3}{5}\right)^{(n+2)\nu} [w]_q 20^{(n+2)q} w(Q_r(y,s)).
\end{aligned}$$

Thus, combining (3.2.20) with this inequality, we finally obtain

$$\frac{w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\} \cap Q_r(y,s)\right)}{w\left(Q_r(y,s)\right)} < \epsilon,$$

which is a contradiction to (3.2.18). This completes the proof. $\square$

According to Lemma 3.2.9, we obtain the power decay estimate for the weight measure of the upper level set $\{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\}$ as follows.

Lemma 3.2.15. *Let w be an A_q weight in $\mathbb{R}^{n+1}$ for some q with $1 < q < \infty$ and let N_1 be given by Lemma 3.2.14. For any $\epsilon > 0$, there exists $\delta =$*

$\delta(\epsilon,\Lambda,n,w,q)>0$ *such that if* $u\in W_2^{2,1}(\Omega_T)$ *is a solution of*

$$\begin{cases} u_t - a_{ij}D_{ij}u &= f \quad \textit{in} \quad \Omega_T \supset Q_6^+, \\ u &= 0 \quad \textit{on} \quad \partial_p\Omega_T \supset T_6, \end{cases}$$

with

$$w\left(\{(x,t)\in\Omega_T : \mathcal{M}(|u_t|^2+|D^2u|^2)(x,t) > N_1^2\}\cap Q_1^+\right) < \epsilon w(Q_1^+), \tag{3.2.21}$$

and if $\mathbf{A}$ *is uniformly parabolic and* $(\delta,6)$*-vanishing, then*

$$\begin{aligned} & w\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|u_t|^2+|D^2u|^2)(x,t) > N_1^{2k}\}\right) \\ & \le \epsilon_1^k w\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|u_t|^2+|D^2u|^2)(x,t) > 1\}\right) \\ & \quad + \sum_{i=1}^k \epsilon_1^i w\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|f|^2)(x,t) > \delta^2 N_1^{2(k-i)}\}\right), \end{aligned}$$

where $\epsilon_1 := 2^q 10^{(n+2)q}\epsilon[w]_q^2$.

Proof. In order to apply Lemma 3.2.9, we first set

$$\begin{aligned} E &:= \{(x,t)\in Q_1^+ : \mathcal{M}(|u_t|^2+|D^2u|^2)(x,t) > N_1^2\} \quad \text{and} \\ F &:= \{(x,t)\in Q_1^+ : \mathcal{M}(|u_t|^2+|D^2u|^2)(x,t) > 1\} \\ & \qquad \cup\{(x,t)\in Q_1^+ : \mathcal{M}(|f|^2)(x,t) > \delta^2\}. \end{aligned}$$

In view of (3.2.21) and Lemma 3.2.14, one can readily check that all the hypotheses of Lemma 3.2.9 are satisfied. Then it follows from Lemma 3.2.9 that $w(E)\le\epsilon_1 w(F)$ for $\epsilon_1 := 2^q 10^{(n+2)q}\epsilon[w]_q^2$. In other words,

$$\begin{aligned} & w\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|u_t|^2+|D^2u|^2)(x,t) > N_1^2\}\right) \\ & \le \epsilon_1 w\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|u_t|^2+|D^2u|^2)(x,t) > 1\}\right) \\ & \quad + \epsilon_1 w\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|f|^2)(x,t) > \delta^2\}\right). \end{aligned}$$

It is clear that for any $k\ge 2$, we have

$$E_k := \{(x,t)\in Q_1^+ : \mathcal{M}(|u_t|^2+|D^2u|^2)(x,t) > N_1^k\} \subset E,$$

and so $w(E_k) < \epsilon w(Q_1^+)$.

Then for each $\lambda := N_1^{k-1}$, we observe that $u_\lambda := \frac{u}{\lambda} \in W_2^{2,1}(\Omega_T)$ is a

solution of

$$\begin{cases} (u_\lambda)_t - a_{ij}D_{ij}u_\lambda &= f_\lambda \quad \text{in} \quad \Omega_T \supset Q_6^+, \\ u_\lambda &= 0 \quad \text{on} \quad \partial_p\Omega_T \supset T_6, \end{cases}$$

with $w(E^\lambda) < \epsilon w(Q_1^+)$, where $f_\lambda := \frac{f}{\lambda}$ and

$$E^\lambda := \{(x,t) \in Q_1^+ : \mathcal{M}(|(u_\lambda)_t|^2 + |D^2u_\lambda|^2)(x,t) > N_1^2\}.$$

Hence, we have

$$\begin{aligned} & w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|(u_\lambda)_t|^2 + |D^2u_\lambda|^2)(x,t) > N_1^2\}\right) \\ & \le \epsilon_1 w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|(u_\lambda)_t|^2 + |D^2u_\lambda|^2)(x,t) > 1\}\right) \\ & \quad + \epsilon_1 w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|f_\lambda|^2)(x,t) > \delta^2\}\right), \end{aligned}$$

which implies

$$\begin{aligned} & w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^2\lambda^2\}\right) \\ & \le \epsilon_1 w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > \lambda^2\}\right) \\ & \quad + \epsilon_1 w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|f|^2)(x,t) > \delta^2\lambda^2\}\right). \end{aligned}$$

Thus, by iterating the above estimate, we conclude that

$$\begin{aligned} & w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > N_1^{2k}\}\right) \\ & \le \epsilon_1^k w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|u_t|^2 + |D^2u|^2)(x,t) > 1\}\right) \\ & \quad + \sum_{i=1}^{k} \epsilon_1^i w\left(\{(x,t) \in Q_1^+ : \mathcal{M}(|f|^2)(x,t) > \delta^2 N_1^{2(k-i)}\}\right), \end{aligned}$$

for any positive integer k. □

We are now ready to prove main result in this section. As mentioned before, we only establish the boundary estimates (3.2.12) in Theorem 3.2.11. Obviously, N_1, ϵ and corresponding δ can be taken to be the same as in the previous lemma.

Proof of (ii) in Theorem 3.2.11. In this proof, c denotes a universal constant that can be computed in terms of Λ, n, Φ and w, and that may vary from line to line.

Applying Lemma 3.2.2 and (3.2.3), hypotheses $|f|^2 \in L_w^\Phi(\Omega_T)$ and $w \in$

$A_{i(\Phi)}$ allow us to find that

$$\begin{aligned}\int_{\Omega_T}|f|^2 &\le \left(\int_{\Omega_T}|f|^{2(i(\Phi)-\epsilon_0)}w\right)^{\frac{1}{i(\Phi)-\epsilon_0}}\left(\int_{Q_{R_0}}w^{\frac{-1}{i(\Phi)-\epsilon_0-1}}\right)^{\frac{i(\Phi)-\epsilon_0-1}{i(\Phi)-\epsilon_0}}\\
&\le \left(\int_{\Omega_T}|f|^{2(i(\Phi)-\epsilon_0)}w\right)^{\frac{1}{i(\Phi)-\epsilon_0}}\left(\frac{[w]_{i(\Phi)-\epsilon_0}}{w(Q_{R_0})}\right)^{\frac{1}{i(\Phi)-\epsilon_0}}|Q_{R_0}|\\
&\le \left(\int_{\Omega_T}\left(|f|^2+1\right)^{(i(\Phi)-\epsilon_0)}w\right)^{\frac{1}{i(\Phi)-\epsilon_0}}\left(\frac{[w]_{i(\Phi)-\epsilon_0}}{w(Q_{R_0})}\right)^{\frac{1}{i(\Phi)-\epsilon_0}}|Q_{R_0}|\\
&\le \left(\frac{c}{\Phi(1)}\int_{\Omega_T}\Phi\left(|f|^2+1\right)w\right)^{\frac{1}{i(\Phi)-\epsilon_0}}\left(\frac{[w]_{i(\Phi)-\epsilon_0}}{w(Q_{R_0})}\right)^{\frac{1}{i(\Phi)-\epsilon_0}}|Q_{R_0}|\\
&\le c\left(\int_{\Omega_T}\Phi\left(|f|^2\right)wdxdt+1\right)^{\frac{1}{i(\Phi)-\epsilon_0}} \le c\left(\||f|^2\|^{\tilde{\beta}}_{L^\Phi_w(\Omega_T)}+1\right), \quad (3.2.22)\end{aligned}$$

for some constants c and $\tilde{\beta}$ depending only on n, Φ and w, where $R_0 > 0$ is a large constant so that $\Omega_T \subset Q_{R_0}$. Hence, we have $|f| \in L^2(\Omega_T)$. Then we recall Lemma 3.2.10 to obtain that there is a solution $u \in W^{2,1}_2(Q^+_{6\rho})$ of (3.2.11) with estimate

$$\|u_t\|_{L^2(Q^+_\rho)} + \|D^2u\|_{L^2(Q^+_\rho)} \le c\left(\|f\|_{L^2(Q^+_{6\rho})} + \frac{1}{\rho^2}\|u\|_{L^2(Q^+_{6\rho})}\right),$$

for some constant $c > 0$ being independent of u and f.

Let us now consider

$$\tilde{u} := \frac{\delta u}{\left(\||f|^2\|_{L^\Phi_w(Q^+_{6\rho})} + \frac{1}{\rho^4}\|u\|^2_{L^2(Q^+_{6\rho})}\right)^{\frac{1}{2}}} \qquad (3.2.23)$$

and

$$\tilde{f} := \frac{\delta f}{\left(\||f|^2\|_{L^\Phi_w(Q^+_{6\rho})} + \frac{1}{\rho^4}\|u\|^2_{L^2(Q^+_{6\rho})}\right)^{\frac{1}{2}}}, \qquad (3.2.24)$$

and then define $\tilde{a}_{ij}(x,t) = a_{ij}(\rho x, \rho^2 t)$, $\tilde{u}_\rho(x,t) = \frac{1}{\rho^2}\tilde{u}(\rho x, \rho^2 t)$, $\tilde{f}_\rho(x,t) = \tilde{f}(\rho x, \rho^2 t)$, $\tilde{w}(x,t) = w(\rho x, \rho^2 t)$ and $\tilde{\Omega}_T = \{\left(\frac{1}{\rho}x, \frac{1}{\rho^2}t\right) : (x,t) \in \Omega_T\}$. Note that $\tilde{u}_\rho \in W_2^{2,1}(\tilde{\Omega}_T)$ is a solution of

$$\begin{cases} (\tilde{u}_\rho)_t - a_{ij}D_{ij}\tilde{u}_\rho = \tilde{f}_\rho & \text{in} \quad \tilde{\Omega}_T \supset Q_6^+, \\ \tilde{u}_\rho = 0 & \text{on} \quad \partial_p\tilde{\Omega}_T \supset T_6, \end{cases}$$

with estimate

$$\|(\tilde{u}_\rho)_t\|_{L^2(Q_1^+)} + \|D^2\tilde{u}_\rho\|_{L^2(Q_1^+)} \leq c\left(\|\tilde{f}_\rho\|_{L^2(Q_6^+)} + \|\tilde{u}_\rho\|_{L^2(Q_6^+)}\right), \tag{3.2.25}$$

where the constant $c > 0$ is independent of u and f.

In a manner similar to that used in (3.2.22), we can estimate

$$\begin{aligned}
&\|\tilde{f}_\rho\|^2_{L^2(Q_6^+)} + \|\tilde{u}_\rho\|^2_{L^2(Q_6^+)} \\
&= \frac{1}{\rho^{n+2}}\left(\int_{Q_{6\rho}^+} \frac{\delta^2|f|^2}{\left(\||f|^2\|_{L_w^\Phi(Q_{6\rho}^+)} + \frac{1}{\rho^4}\|u\|^2_{L^2(Q_{6\rho}^+)}\right)} dxdt \right.\\
&\qquad \left. + \frac{1}{\rho^4}\int_{Q_{6\rho}^+} \frac{\delta^2|u|^2}{\left(\||f|^2\|_{L_w^\Phi(Q_{6\rho}^+)} + \frac{1}{\rho^4}\|u\|^2_{L^2(Q_{6\rho}^+)}\right)} dxdt\right) \\
&\leq c\delta^2\left(\left\|\frac{|f|^2}{\left(\||f|^2\|_{L_w^\Phi(Q_{6\rho}^+)} + \frac{1}{\rho^4}\|u\|^2_{L^2(Q_{6\rho}^+)}\right)}\right\|^{\tilde{\beta}}_{L_w^\Phi(Q_{6\rho}^+)} + 1\right) + c\delta^2 \\
&\leq c\delta^2.
\end{aligned} \tag{3.2.26}$$

Using weak 1-1 estimate (3.2.6), along with (3.2.25) and (3.2.26), we deduce that

$$\begin{aligned}
&\frac{1}{|Q_1^+|}|\{(x,t) \in Q_1^+ : \mathcal{M}(|(\tilde{u}_\rho)_t|^2 + |D^2\tilde{u}_\rho|^2)(x,t) > N_1^2\}| \\
&\leq c\fint_{Q_1^+} |(\tilde{u}_\rho)_t|^2 + |D^2\tilde{u}_\rho|^2 dxdt \\
&\leq c\left(\fint_{Q_6^+} |\tilde{f}_\rho|^2 dxdt + \fint_{Q_6^+} |\tilde{u}_\rho|^2 dxdt\right) \leq c\delta^2.
\end{aligned}$$

Then Lemma 3.2.1 yields

$$
\begin{aligned}
&\frac{1}{\tilde{w}(Q_1^+)}\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|(\tilde{u}_\rho)_t|^2+|D^2\tilde{u}_\rho|^2)(x,t)>N_1^2\}\right)\\
&<\beta\left(\frac{|\{(x,t)\in Q_1^+ : \mathcal{M}(|(\tilde{u}_\rho)_t|^2+|D^2\tilde{u}_\rho|^2)(x,t)>N_1^2\}|}{|Q_1^+|}\right)^\nu\\
&\le c\beta\delta^{2\nu}<\epsilon,
\end{aligned}
$$

by taking δ sufficiently small so that the last inequality holds. Therefore all the hypotheses of Lemma 3.2.15 are satisfied, and so Lemma 3.2.15 allows us to find that

$$
\begin{aligned}
&\sum_{k=1}^\infty \Phi(N_1^{2k})\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|(\tilde{u}_\rho)_t|^2+|D^2\tilde{u}_\rho|^2)(x,t)>N_1^{2k}\}\right) \qquad (3.2.27)\\
&\le \sum_{k=1}^\infty \Phi(N_1^{2k})\Bigg\{\epsilon_1^k\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|(\tilde{u}_\rho)_t|^2+|D^2\tilde{u}_\rho|^2)(x,t)>1\}\right)\\
&\qquad\qquad +\sum_{i=1}^k \epsilon_1^i\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|\tilde{f}_\rho|^2)(x,t)>\delta^2N_1^{2(k-i)}\}\right)\Bigg\}.
\end{aligned}
$$

From $\Phi\in\Delta_2\cap\nabla_2$, one can easily check that $\Phi(N_1^2)\le\mu\Phi(1)$ for some constant $\mu>1$ that depends on $\lambda=N_1^2$. Iterating this inequality, we see $\Phi(N_1^{2k})\le\mu^k\Phi(1)$, and so we get

$$
\begin{aligned}
&\sum_{k=1}^\infty \Phi(N_1^{2k})\epsilon_1^k\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|(\tilde{u}_\rho)_t|^2+|D^2\tilde{u}_\rho|^2)(x,t)>1\}\right)\\
&\le \Phi(1)\tilde{w}(Q_1^+)\sum_{k=1}^\infty(\mu\epsilon_1)^k. \qquad (3.2.28)
\end{aligned}
$$

Analogously, since $\Phi(N_1^{2k})\le\mu^i\Phi(N_1^{2(k-i)})$, we can deduce from Lemma 3.2.6 and Lemma 3.2.7 that

$$
\begin{aligned}
&\sum_{k=1}^\infty \Phi(N_1^{2k})\left(\sum_{i=1}^k \epsilon_1^i\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|\tilde{f}_\rho|^2)(x,t)>\delta^2N_1^{2(k-i)}\}\right)\right)\\
&\le \sum_{i=1}^\infty(\mu\epsilon_1)^i\sum_{k=i}^\infty \Phi(N_1^{2(k-i)})\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|\tilde{f}_\rho|^2)(x,t)>\delta^2N_1^{2(k-i)}\}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{\infty}(\mu\epsilon_1)^i \sum_{k=i}^{\infty}\Phi(N_1^{2(k-i)})\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}\left(\frac{|\tilde{f}_\rho|^2}{\delta^2}\right)(x,t) > N_1^{2(k-i)}\}\right)\\
&\leq c\sum_{i=1}^{\infty}(\mu\epsilon_1)^i \int_{Q_1^+}\Phi\left(\mathcal{M}\left(\frac{|\tilde{f}_\rho|^2}{\delta^2}\right)(x,t)\right)\tilde{w}(x,t)dxdt\\
&\leq c\sum_{i=1}^{\infty}(\mu\epsilon_1)^i \int_{Q_1^+}\Phi\left(\frac{|\tilde{f}_\rho|^2}{\delta^2}\right)\tilde{w}(x,t)dxdt\\
&= c\sum_{i=1}^{\infty}(\mu\epsilon_1)^i \frac{1}{\rho^{n+2}}\int_{Q_\rho^+}\Phi\left(\frac{|\tilde{f}|^2}{\delta^2}\right)w(x,t)dxdt\\
&\leq c\sum_{i=1}^{\infty}(\mu\epsilon_1)^i\left(\left\|\frac{|\tilde{f}|^2}{\delta^2}\right\|_{L_w^{\Phi}(Q_\rho^+)}^{\tilde{\beta}}+1\right)\leq c\sum_{i=1}^{\infty}(\mu\epsilon_1)^i, \qquad (3.2.29)
\end{aligned}
$$

where the last inequality holds because of (3.2.24).

Inserting (3.2.28) and (3.2.29) into (3.2.27), we conclude that

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\Phi(N_1^{2k})\tilde{w}\left(\{(x,t)\in Q_1^+ : \mathcal{M}(|(\tilde{u}_\rho)_t|^2+|D^2\tilde{u}_\rho|^2)(x,t) > N_1^{2k}\}\right)\\
&\leq c\sum_{k=1}^{\infty}(\mu\epsilon_1)^k \leq c,
\end{aligned}
$$

by taking ϵ_1 so that $\mu\epsilon_1 < 1$. Therefore, it follows from Lemmas 3.2.6 and 3.2.7 that

$$
\begin{aligned}
&\int_{Q_1^+}\Phi\left(|(\tilde{u}_\rho)_t|^2\right)\tilde{w}(x,t)dxdt + \int_{Q_1^+}\Phi\left(|D^2\tilde{u}_\rho|^2\right)\tilde{w}(x,t)dxdt\\
&\leq c\int_{Q_1^+}\Phi\left(\mathcal{M}\left(|(\tilde{u}_\rho)_t|^2+|D^2\tilde{u}_\rho|^2\right)\right)\tilde{w}(x,t)dxdt \leq c,
\end{aligned}
$$

and this implies

$$
\int_{Q_\rho^+}\Phi\left(|\tilde{u}_t|^2\right)w(x,t)dxdt + \int_{Q_\rho^+}\Phi\left(|D^2\tilde{u}|^2\right)w(x,t)dxdt \leq c.
$$

Furthermore, inequality (3.2.5) yields

$$
\left\||\tilde{u}_t|^2\right\|_{L_w^{\Phi}(Q_\rho^+)} + \left\||D^2\tilde{u}|^2\right\|_{L_w^{\Phi}(Q_\rho^+)} \leq c,
$$

for some positive constant $c = c(\Lambda, n, \Phi, w)$. Hence, recalling the definition of $\tilde{u}$ in (3.2.23), we finally obtain the desired estimates (3.2.12). $\square$

3.2.4 Global weighted Orlicz estimates

In this section, we prove one of the main results in this chapter. The key idea of the proof is to use standard covering and flattening arguments in order to derive the weighted Orlicz estimates from the interior and boundary estimates that were established in the previous sections, under the *a priori* assumption which can be removed by means of a suitable approximation procedure. Hereafter, c will denote a universal constant, depending only on n, Λ, w, Φ and $\partial\Omega$, that may vary from line to line.

Proof of Theorem 3.2.4. First, we assume

$$|u|^2, |Du|^2 \in L^{\Phi}_{w}(\Omega_T) \tag{3.2.30}$$

which will be eliminated later in the proof. Let us fix any point $x_0 \in \partial\Omega$. Given hypothesis $\partial\Omega \in C^{1,1}$, we may assume that

$$\Omega \cap B_r(x_0) = \{x \in \Omega : x_n > \gamma(x')\} \cap B_r(x_0)$$

for some small $r > 0$ and some $C^{1,1}$ function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\bigtriangledown\gamma(x'_0) = 0$ and $\| \bigtriangledown^2 \gamma\|_{L^\infty(\mathbb{R}^{n-1})} < \infty$. In order to flatten out the boundary near x_0, we use a change of variables. More precisely, we define

$$\begin{cases} y_i &= & x_i & =: \phi^i(x), \quad \text{if } i = 1, 2, \cdots, n-1, \\ y_n &= & x_n - \gamma(x') & =: \phi^n(x), \end{cases}$$

and write $y = \phi(x)$. Set $\phi := \psi^{-1}$ and write $x = \psi(y)$. Select $\rho > 0$ to be so small that the half ball $B^+_{12\rho} \subset \phi(\Omega \cap B_r(x_0))$. We now consider $\tilde{u}(y, t) = u(\psi(y), t) = u(x, t)$ for $y \in B^+_{6\rho}$ and $\tilde{w}(y, t) = w(\psi(y), t)$ for $y \in \mathbb{R}^n$. Then it is straightforward to check that $\tilde{w} \in A_{i(\Phi)}$ and $\tilde{u} \in W^{2,1}_2(Q^+_{6\rho})$ is a solution of

$$\begin{cases} \tilde{u}_t - \tilde{a}_{lm} D_{y_l y_m}\tilde{u} &= & \tilde{f} & \text{in} & Q^+_{6\rho}, \\ \tilde{u} &= & 0 & \text{on} & T_{6\rho}, \end{cases} \tag{3.2.31}$$

where

$$\begin{aligned} \tilde{a}_{lm}(y, t) &= a_{ij}(\psi(y), t)\phi^l_{x_i}(\psi(y), t)\phi^m_{x_j}(\psi(y), t), \text{ and} \\ \tilde{f}(y, t) &= f(\psi(y), t) - a_{ij}(\psi(y), t)\phi^l_{x_i x_j}(\psi(y), t) D_{y_l}\tilde{u}. \end{aligned}$$

Recalling the hypotheses on $\mathbf{A}$ and $\partial\Omega$, we notice

$$\left\| |\tilde{f}|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{6\rho})} \leq \left\| |f(\psi(y),t)|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{6\rho})} + c \left\| |D\tilde{u}|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{6\rho})}, \qquad (3.2.32)$$

and hence, from *a priori* assumption (3.2.30) and the hypothesis on f, we have $|\tilde{f}|^2 \in L^{\Phi}_{\tilde{w}}(Q^+_{6\rho})$. Moreover, the resulting matrix of coefficients

$$\tilde{\mathbf{A}}(y,t) = (\tilde{a}_{lm}(y,t)) = [\triangledown \phi(\psi(y),t)] \cdot \mathbf{A}(\psi(y),t) \cdot [\triangledown \phi(\psi(y),t)]^t$$

satisfies small BMO assumption (3.0.3). Indeed, from the conditions on $\mathbf{A}$ and $\partial\Omega$, a direct computation gives

$$\begin{aligned} \|\tilde{\mathbf{A}}\|_* &\leq c\left(\|\mathbf{A}\|_* + \| \triangledown \gamma\|_{L^\infty(B'_r(x'_0))} + \| \triangledown \gamma\|^2_{L^\infty(B'_r(x'_0))} \right) \\ &\leq c\left(\delta + r\| \triangledown^2 \gamma\|_{L^\infty(B'_r(x'_0))} + r^2\| \triangledown^2 \gamma\|^2_{L^\infty(B'_r(x'_0))} \right) \\ &\leq c(\delta + r + r^2). \end{aligned}$$

We choose $\delta = \delta(n,\Lambda,\gamma) > 0$ and $r = r(n,\Lambda,\gamma) > 0$ so that all the hypotheses of Theorem 3.2.11 are satisfied, and then we recall Theorem 3.2.11 to discover that $|\tilde{u}_t|^2$, $|D^2\tilde{u}|^2 \in L^{\Phi}_{\tilde{w}}(Q^+_{\rho})$ with estimate

$$\left\| |\tilde{u}_t|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{\rho})} + \left\| |D^2\tilde{u}|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{\rho})} \leq c\left(\left\| |\tilde{f}|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{6\rho})} + \frac{1}{\rho^4}\|\tilde{u}\|^2_{L^2(Q^+_{6\rho})} \right),$$

for some constant $c > 0$ being independent of $\tilde{u}$ and $\tilde{f}$. Then it follows from (3.2.32) that

$$\begin{aligned} &\left\| |\tilde{u}_t|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{\rho})} + \left\| |D^2\tilde{u}|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{\rho})} \qquad (3.2.33) \\ &\quad \leq c\left(\left\| |f(\psi(y),t)|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{6\rho})} + \left\| |D\tilde{u}|^2 \right\|_{L^{\Phi}_{\tilde{w}}(Q^+_{6\rho})} + \frac{1}{\rho^4}\|\tilde{u}\|^2_{L^2(Q^+_{6\rho})} \right). \end{aligned}$$

Now let us define $\tilde{\Phi}(\sigma) := \Phi(\sigma^2)$, and then we can easily check that $\tilde{\Phi}$ satisfies all the properties imposed on Φ. In addition, we note that $w \in A_{i(\Phi)} \subset A_{2i(\Phi)} = A_{i(\tilde{\Phi})}$. Then inequality (3.2.33) directly implies that

$$\begin{aligned} &\|\tilde{u}_t\|_{L^{\tilde{\Phi}}_{\tilde{w}}(Q^+_{\rho})} + \|D^2\tilde{u}\|_{L^{\tilde{\Phi}}_{\tilde{w}}(Q^+_{\rho})} \qquad (3.2.34) \\ &\quad \leq c\left(\|f(\psi(y),t)\|_{L^{\tilde{\Phi}}_{\tilde{w}}(Q^+_{6\rho})} + \|D\tilde{u}\|_{L^{\tilde{\Phi}}_{\tilde{w}}(Q^+_{\rho})} + \|\tilde{u}\|^2_{L^2(Q^+_{6\rho})} \right). \end{aligned}$$

Converting back to the original x-variables, we conclude

$$\begin{aligned}
&\|u_t\|_{L^{\tilde{\Phi}}_w(V_\rho)} + \|D^2u\|_{L^{\tilde{\Phi}}_w(V_\rho)} \\
&\leq c\Bigg(\|f(\psi(y),t)\|_{L^{\tilde{\Phi}}_w\left(\psi(B^+_{6\rho})\times(-\rho^2,\rho^2)\right)} + \|Du\|_{L^{\tilde{\Phi}}_w\left(\psi(B^+_{6\rho})\times(-\rho^2,\rho^2)\right)} \\
&\qquad + \|u\|^2_{L^2\left(\psi(B^+_{6\rho})\times(-\rho^2,\rho^2)\right)}\Bigg),
\end{aligned}$$

where $V_\rho := \psi(B^+_\rho) \times \left(-\rho^2, \rho^2\right)$. For the estimates near the corner and on the bottom of Ω_T, we extend u by zero for $t \leq 0$. Since $\partial_p\Omega_T$ is compact, we can cover $\partial_p\Omega_T$ by a finite number of sets $V_{\rho_1}, V_{\rho_2}, \cdots, V_{\rho_N}$ as above and find a finite number of small positive constants $\rho_1, \rho_2, \cdots, \rho_N$. Thus, by summing the resulting estimates, along with interior estimate (3.2.10) over some open set $V_{\rho_0} \subset\subset \Omega_T$ so that $\Omega_T \subset \cup_{i=0}^N V_{\rho_i}$, we obtain that

$$u_t,\ D^2u \in L^{\tilde{\Phi}}_w(\Omega_T)$$

with estimate

$$\|u_t\|_{L^{\tilde{\Phi}}_w(\Omega_T)} + \|D^2u\|_{L^{\tilde{\Phi}}_w(\Omega_T)} \leq c\left(\|f\|_{L^{\tilde{\Phi}}_w(\Omega_T)} + \|Du\|_{L^{\tilde{\Phi}}_w(\Omega_T)} + \|u\|^2_{L^2(\Omega_T)}\right),$$

which implies

$$\|u\|_{W^{2,1}L^{\tilde{\Phi}}_w(\Omega_T)} \leq c\left(\|f\|_{L^{\tilde{\Phi}}_w(\Omega_T)} + \|u\|_{L^{\tilde{\Phi}}_w(\Omega_T)} + \|Du\|_{L^{\tilde{\Phi}}_w(\Omega_T)} + \|u\|^2_{L^2(\Omega_T)}\right). \tag{3.2.35}$$

From the uniqueness of solutions of a homogeneous equation, we finally arrive at the desired estimate

$$\|u\|_{W^{2,1}L^{\tilde{\Phi}}_w(\Omega_T)} \leq c\|f\|_{L^{\tilde{\Phi}}_w(\Omega_T)}. \tag{3.2.36}$$

Indeed, to do this, we argue by contradiction. If estimate (3.2.36) is false, there exist $\{u^k\}_{k=1}^\infty$ and $\{f^k\}_{k=1}^\infty$ such that u^k is a solution of

$$\left\{\begin{array}{rcccc} u^k_t - a_{ij}D_{ij}u^k &=& f^k & \text{in} & \Omega_T, \\ u^k &=& 0 & \text{on} & \partial_p\Omega_T, \end{array}\right. \tag{3.2.37}$$

with

$$\|u^k\|_{W^{2,1}L^{\tilde{\Phi}}_w(\Omega_T)} > k\|f^k\|_{L^{\tilde{\Phi}}_w(\Omega_T)}, \tag{3.2.38}$$

for any $k \geq 1$. Without loss of generality, we may assume

$$\|u^k\|_{W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T)} = 1$$

and then (3.2.38) implies

$$\|f^k\|_{L_w^{\tilde{\Phi}}(\Omega_T)} < \frac{1}{k} \to 0 \quad \text{as } k \to \infty. \tag{3.2.39}$$

In particular, $\{u^k\}_{k=1}^{\infty}$ is uniformly bounded in $W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T)$, and thus, there exist a subsequence, which we still denote by $\{u^k\}_{k=1}^{\infty}$, and a function $u^0 \in W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T)$ such that

$$u^k \rightharpoonup u^0 \quad \text{in } W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T) \quad \text{as } k \to \infty.$$

It is easy to check that u^0 is a solution of

$$\begin{cases} u_t^0 - a_{ij}D_{ij}u^0 = 0 & \text{in} \quad \Omega_T, \\ \qquad\qquad u^0 = 0 & \text{on} \quad \partial_p\Omega_T. \end{cases} \tag{3.2.40}$$

By the uniqueness of solutions to (3.2.40), we must have $u^0 = 0$ in Ω_T.

Moreover, since $L_w^{\tilde{\Phi}}(\Omega_T)$ is continuously embedded into $L^2(\Omega_T)$, inequality (3.2.39) yields

$$f^k \to 0 \text{ in } L^2(\Omega_T) \quad \text{as } k \to \infty. \tag{3.2.41}$$

Recalling the $W_2^{2,1}$ estimates for (3.2.37) (see [9, Theorem 4.3]), we have

$$\|Du^k\|_{L^2(\Omega_T)} \leq c\|f^k\|_{L^2(\Omega_T)}$$

and so it follows from (3.2.41) that

$$Du^k \to 0 \quad \text{in } L^2(\Omega_T) \quad \text{as } k \to \infty.$$

Then letting $\mu := w(x,t)dxdt$, we see that

$$Du^k \to 0 \quad \mu\text{-a.e. in } \Omega_T \quad \text{as } k \to \infty \text{ (up to subsequence)},$$

and so

$$\tilde{\Phi}(|Du^k|) \to 0 \quad \mu\text{-a.e. in } \Omega_T \quad \text{as } k \to \infty.$$

Since Du^k is bounded in $L_w^{\tilde{\Phi}}(\Omega_T)$, we note that $\int_{\Omega_T} \tilde{\Phi}(|Du^k|)w(x,t)dxdt$ is bounded. The Lebesgue Dominated Convergence Theorem gives us to get

$$\int_{\Omega_T} \tilde{\Phi}(|Du^k|)w(x,t)dxdt \to 0 \text{ as } k \to \infty,$$

which implies

$$Du^k \to 0 \text{ in } L_w^{\tilde{\Phi}}(\Omega_T) \text{ as } k \to \infty.$$

However, from (3.2.35), we find that

$$1 \le c\left(\|f^k\|_{L_w^{\tilde{\Phi}}(\Omega_T)} + \|u^k\|_{L_w^{\tilde{\Phi}}(\Omega_T)} + \|Du^k\|_{L_w^{\tilde{\Phi}}(\Omega_T)} + \|u^k\|^2_{L^2(\Omega_T)}\right) \longrightarrow 0$$

as $k \to \infty$, which is a contradiction.

Now it only remains to remove the *a priori* assumption (3.2.30). We first choose a sequence $\{a_{ij}^k\}_{k=1}^{\infty}$ of smooth functions with a uniform (δ, R)-vanishing property such that

$$a_{ij}^k \to a_{ij} \text{ in } L^q(\Omega_T) \text{ as } k \to \infty \text{ for each } 1 < q < \infty. \tag{3.2.42}$$

We also select a sequence $\{f^k\}_{k=1}^{\infty}$ of smooth functions in $C_0^{\infty}(\Omega_T)$ such that $f^k \to f$ in $L_w^{\tilde{\Phi}}(\Omega_T)$ as $k \to \infty$ and

$$\|f^k\|_{L_w^{\tilde{\Phi}}(\Omega_T)} \le \|f\|_{L_w^{\tilde{\Phi}}(\Omega_T)} + 1. \tag{3.2.43}$$

Then recalling [9, Theorem 4.3], we observe that there exists a unique solution $u^k \in W_p^{2,1}(\Omega_T)$ of

$$\left\{\begin{array}{rcccc} u_t^k - a_{ij}^k D_{ij}u^k & = & f^k & \text{in} & \Omega_T, \\ u^k & = & 0 & \text{on} & \partial_p\Omega_T. \end{array}\right. \tag{3.2.44}$$

for any $1 < p < \infty$. Then it is clear that these solutions u^k are in $W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T)$. However, it follows from estimate (3.2.36) that

$$\|u^k\|_{W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T)} \le c\|f^k\|_{L_w^{\tilde{\Phi}}(\Omega_T)}, \tag{3.2.45}$$

where the constant c is independent of k. Therefore, from (3.2.43) and (3.2.45), we see that

$$\|u^k\|_{W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T)} \le c\left(\|f\|_{L_w^{\tilde{\Phi}}(\Omega_T)} + 1\right). \tag{3.2.46}$$

Hence, $\{u^k\}_{k=1}^{\infty}$ is uniformly bounded in $W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T)$, and so there exist a subsequence, which we still denote by $\{u^k\}_{k=1}^{\infty}$, and a function $v \in W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T)$ such that

$$u^k \rightharpoonup v \quad \text{weakly in } W^{2,1}L_w^{\tilde{\Phi}}(\Omega_T) \quad \text{as } k \to \infty. \tag{3.2.47}$$

In view of (3.2.42)-(3.2.44) and (3.2.47), it is easy to check that v is also a solution to (3.0.1). Then the uniqueness of solutions to (3.0.1) asserts $u = v$. Hence, the proof is completed.

□

3.3 Weighted estimates in variable exponent spaces

3.3.1 Assumptions and main result

We consider the *variable exponent* $p(z) = p(x,t) = p(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ with

$$1 < \gamma_1 := \inf_{z \in \mathbb{R}^{n+1}} p(z) \leq \sup_{z \in \mathbb{R}^{n+1}} p(z) =: \gamma_2 < \infty, \tag{3.3.1}$$

for some constants γ_1 and γ_2, and its conjugate exponent $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$. Let $w : \mathbb{R}^{n+1} \to (0,\infty)$ be a locally integrable function, which is called a *weight*. For $U \subset \mathbb{R}^{n+1}$, we define the *weighted variable exponent Lebesgue space* $L^{p(\cdot)}(U,w)$ to be the set of all measurable functions $g : U \to \mathbb{R}$ such that the *modular*

$$\varrho_{p(\cdot),w}(g) := \int_U |g|^{p(z)} w(z)\, dz$$

is finite. Then $L^{p(\cdot)}(U,w)$ becomes a Banach space equipped with the following *Luxemburg norm*:

$$\|g\|_{L^{p(\cdot)}(U,w)} := \inf \left\{ \lambda > 0 : \varrho_{p(\cdot),w}\left(\frac{g}{\lambda}\right) \leq 1 \right\}. \tag{3.3.2}$$

If $w \equiv 1$, we simply write $L^{p(\cdot)}(U) = L^{p(\cdot)}(U,1)$, which is the usual variable exponent Lebesgue space. On the other hand, if the variable exponent $p(\cdot)$ is constant, i.e. $p(\cdot) \equiv p$, then the space $L^{p(\cdot)}(U,w)$ coincides with the weighted Lebesgue space $L^p(U,w)$, i.e. its norm becomes the classical norm of the space $L^p(U,w)$ as follows:

$$\|g\|_{L^p(U,w)} = \left(\int_U |g|^p w(z)\, dz \right)^{\frac{1}{p}}.$$

We now present crucial conditions on the variable exponent $p(\cdot)$ and the weight w.

Definition 3.3.1. We say that $p(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ is *log-Hölder continuous*, denoted by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^{n+1})$, if

$$|p(\xi) - p(\tilde{\xi})| \le \frac{c_{LH}}{\log\left(e + 1/|\xi - \tilde{\xi}|\right)} \tag{3.3.3}$$

and

$$|p(\xi) - p_\infty| \le \frac{c_{LH}}{\log(e + |\xi|)},$$

for all $\xi, \tilde{\xi} \in \mathbb{R}^{n+1}$ and for some $p_\infty \in \mathbb{R}$ and $c_{LH} = c_{LH}(p(\cdot)) > 0$. Here, c_{LH} is called the *log-Hölder constant* of $p(\cdot)$.

In particular, if $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^{n+1})$ satisfies (3.3.1), we write $p(\cdot) \in \mathcal{P}^{\log}_{\pm}(\mathbb{R}^{n+1})$.

Hereafter, we abbreviate $\mathcal{P}^{\log}_{\pm} := \mathcal{P}^{\log}_{\pm}(\mathbb{R}^{n+1})$ for the sake of simplicity. We point out that the condition (3.3.3) implies that

$$\theta(r) \log\left(\frac{1}{r}\right) \le M, \quad \text{for all } 0 < r < \infty, \tag{3.3.4}$$

where $\theta(\cdot) : [0, \infty) \to [0, 2\gamma_2]$ with $\theta(0) = 0$ is the modulus of continuity of $p(\cdot)$ with respect to the parabolic distance d_p such that

$$\theta(r) := \sup\left\{|p(\xi) - p(\tilde{\xi})| : d_p(\xi, \tilde{\xi}) \le r \text{ and } \xi, \tilde{\xi} \in \mathbb{R}^{n+1}\right\} \tag{3.3.5}$$

and the constant $M > 0$ depends only on c_{LH} and γ_2. Indeed, if $d_p(\xi, \tilde{\xi}) = \tilde{r} \le r \le 1$, we have $|\xi - \tilde{\xi}| \le \tilde{r}\sqrt{1 + \tilde{r}^2} \le \sqrt{2}\tilde{r}$. Then a direct computation yields

$$\begin{aligned} |p(\xi) - p(\tilde{\xi})| \log\left(\frac{1}{r}\right) &\le |p(\xi) - p(\tilde{\xi})| \log\left(\frac{1}{\tilde{r}}\right) \le |p(\xi) - p(\tilde{\xi})| \log\left(\frac{\sqrt{2}}{|\xi - \tilde{\xi}|}\right) \\ &\le |p(\xi) - p(\tilde{\xi})| \log\left(e + \frac{1}{|\xi - \tilde{\xi}|}\right) + |p(\xi) - p(\tilde{\xi})| \log\sqrt{2} \end{aligned}$$

which implies that

$$\theta(r) \log\left(\frac{1}{r}\right) \le c_{LH} + 2\gamma_2 \log\sqrt{2} \ \text{ for all } 0 < r \le 1.$$

In addition, from (3.3.5) we have

$$|p(\xi) - p(\tilde{\xi})| \leq \theta(d_p(\xi, \tilde{\xi})). \tag{3.3.6}$$

Definition 3.3.2. For $U \subset \mathbb{R}^{n+1}$, we say that the weight w is of $A_{p(\cdot)}(U)$ *class*, denoted by $w \in A_{p(\cdot)}(U)$, if

$$[w]_{A_{p(\cdot)}(U)} := \sup_{C \subset U} |C|^{-p_C} \|w\|_{L^1(C)} \|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)} < \infty,$$

where C is any parabolic cube and p_C is the harmonic average of $p(\cdot)$ over C denoted by

$$p_C := \left(\fint_C p(z)^{-1} \, dz \right)^{-1}.$$

In particular, when $U = \mathbb{R}^{n+1}$, we simply write $A_{p(\cdot)} = A_{p(\cdot)}(\mathbb{R}^{n+1})$.

Here, $[w]_{A_{p(\cdot)}(U)}$ is called the $A_{p(\cdot)}$-constant of w and $\|\cdot\|_{L^{p'(\cdot)/p(\cdot)}(C)}$ is defined by (3.3.2) with $p(\cdot)$ replaced by $p'(\cdot)/p(\cdot)$. Note that $p'(\cdot)/p(\cdot)$ might be less than one and, in this case, $\|\cdot\|_{L^{p'(\cdot)/p(\cdot)}(C)}$ is not a norm but it is only a quasi-norm. When $p(\cdot)$ is constant, i.e. $p(\cdot) \equiv p$, the $A_{p(\cdot)}(U)$ class is the ordinary Muckenhoupt class $A_p(U)$ and we have

$$[w]_{A_p(U)} = \sup_{C \subset U} \left(\fint_C w(z) \, dz \right) \left(\fint_C w(z)^{-\frac{1}{p-1}} \, dz \right)^{p-1},$$

which is the classical definition of the A_p-constant of w.

Remark 3.3.3. We adopted parabolic cubes, instead of usual cubes, in the definition of $A_{p(\cdot)}$ class. This is suitable for our problem that is dealing with parabolic equations. We also note that the weight $w \in A_{p(\cdot)}$ satisfies the doubling property as the same as the classical Muckenhoupt weight. On the other hand, we still used the Euclidean distance in the definition of log-Hölder continuity.

We suppose $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$ and $w \in A_{p(\cdot)}$, and recall the bounded domain $\Omega_T = \Omega \times (0, T]$. The *parabolic weighted variable exponent Sobolev space* $W^{2,1}_{p(\cdot)}(\Omega_T, w)$ is defined as

$$W^{2,1}_{p(\cdot)}(\Omega_T, w) := \left\{ g \in L^{p(\cdot)}(\Omega_T, w) : |Dg|, |D^2 g|, g_t \in L^{p(\cdot)}(\Omega_T, w) \right\},$$

endowed with the norm

$$\begin{aligned}\|g\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} &= \|g\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Dg\|_{L^{p(\cdot)}(\Omega_T,w)} \\ &\quad + \|D^2 g\|_{L^{p(\cdot)}(\Omega_T,w)} + \|g_t\|_{L^{p(\cdot)}(\Omega_T,w)}\end{aligned}$$

where we abbreviate

$$\|Dg\|_{L^{p(\cdot)}(\Omega_T,w)} := \||Dg|\|_{L^{p(\cdot)}(\Omega_T,w)},\ \|D^2 g\|_{L^{p(\cdot)}(\Omega_T,w)} := \||D^2 g|\|_{L^{p(\cdot)}(\Omega_T,w)}$$

for the sake of convenience. We also define $W^{2,1}_{p(\cdot)}(Q_r(\xi),w)$ and $W^{2,1}_{p(\cdot)}(Q_r^+,w)$ for parabolic cylinders $Q_r(\xi)$ and Q_r^+ in the same way. In addition, we denote

$$\overset{\circ}{W}{}^{2,1}_{p(\cdot)}(\Omega_T,w) = \left\{ g \in W^{2,1}_{p(\cdot)}(\Omega_T,w) : g = 0 \text{ on } \partial_p \Omega_T \right\}.$$

We remark that the log-Hölder continuity is considered as an unavoidable condition, because given the variable exponent $p(\cdot)$ with this condition, the properties of the classical Lebesgue and Sobolev spaces, such as Sobolev embeddings, Poincaré's inequality and the boundedness of singular integral operators are valid in variable exponent Lebesgue and Sobolev spaces. We further discuss on weighted variable exponent spaces in the next section.

Our main result in this chapter is the following:

Theorem 3.3.4. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$ with the log-Hölder constant c_{LH} and the modulus of continuity $\theta(\cdot)$, and $w \in A_{p(\cdot)}$. Assume $\partial\Omega \in C^{1,1}$ and $f \in L^{p(\cdot)}(\Omega_T,w)$. Then there is a small $\delta = \delta(n,\Lambda,\gamma_1,\gamma_2,c_{LH},w,\partial\Omega) > 0$ so that if $\mathbf{A}$ is (δ,R)-vanishing for some $R > 0$, the problem (3.0.1) has a unique strong solution $u \in \overset{\circ}{W}{}^{2,1}_{p(\cdot)}(\Omega_T,w)$ and we have the estimate*

$$\|u\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \le c\|f\|_{L^{p(\cdot)}(\Omega_T,w)} \tag{3.3.7}$$

for some positive constant $c = c(n,\Lambda,\gamma_1,\gamma_2,c_{LH},\theta(\cdot),w,R,\Omega,T)$.

Thanks to the linearity of the equation (3.0.1), we have a direct consequence of the above theorem as follows.

Corollary 3.3.5. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$ with the log-Hölder constant c_{LH} and the modulus of continuity $\theta(\cdot)$, and $w \in A_{p(\cdot)}$. Assume $\partial\Omega \in C^{1,1}$, $f \in L^{p(\cdot)}(\Omega_T,w)$ and $\phi \in W^{2,1}_{p(\cdot)}(\Omega_T,w)$. Then there is a small $\delta = \delta(n,\Lambda,\gamma_1,\gamma_2,$ $c_{LH},[w]_{A_{p(\cdot)}},\partial\Omega) > 0$ so that if $\mathbf{A}$ is (δ,R)-vanishing for some $R > 0$, the* *problem*

$$\begin{cases} u_t - a_{ij}D_{ij}u &= f \quad in \quad \Omega_T, \\ u &= \phi \quad on \quad \partial_p\Omega_T \end{cases}$$

has a unique solution $u \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$ *with* $u - \phi \in \mathring{W}^{2,1}_{p(\cdot)}(\Omega_T, w)$, *and we have the estimate*

$$\|u\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \le c\left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|\phi\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)}\right) \tag{3.3.8}$$

for some positive constant $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, w, \Omega, R, T)$.

3.3.2 Preliminaries

We introduce the properties of weights belonging to A_p class for $1 < p < \infty$. For their proofs, we refer to [20, 40, 68]. Let us define

$$w(E) := \int_E w(z)\,dz,$$

for a measurable set $E \subset \mathbb{R}^{n+1}$, and let $U \subset \mathbb{R}^{n+1}$ be an open set. We first remark that $u \in A_p(U)$ if and only if there exists $c \ge 1$ such that

$$\left(\fint_C g\,dz\right)^p \le \frac{c}{w(C)}\int_C g^p w(z)\,dz, \tag{3.3.9}$$

for all nonnegative measurable functions g and all parabolic cubes $C \subset U$. In particular, the smallest constant c satisfying the inequality (3.3.9) is equal to $[w]_{A_p(U)}$.

Lemma 3.3.6. *Let* $w \in A_p(U)$ *for some* $1 < p < \infty$.

(1) *There exist positive constants* ν_1 *and* $d_1 \ge 1$ *depending only on* n, p *and* $[w]_{A_p(U)}$ *such that*

$$\left(\fint_C w(z)^{1+\nu_1}\,dz\right)^{\frac{1}{1+\nu_1}} \le d_1 \fint_C w(z)\,dz$$

for all parabolic cubes $C \subset U$.

(2) *We have*

$$\frac{1}{[w]_{A_p(U)}}\left(\frac{|E|}{|C|}\right)^p \le \frac{w(E)}{w(C)} \le d_1\left(\frac{|E|}{|C|}\right)^{\frac{\nu_1}{1+\nu_1}}$$

for all parabolic cubes $C \subset U$ and all measurable subsets E of C, where ν_1 and d_1 have been determined in (1).

(3) *There exist $\epsilon_1 \in (0, p-1)$ and $\tilde{d}_1 \geq 1$ depending only on n, p and $[w]_{A_p(U)}$ such that $w \in A_{p-\epsilon_1}(U)$ with $[w]_{A_{p-\epsilon_1}(U)} \leq \tilde{d}_1$.*

Remark 3.3.7. In view of the proofs of Theorem 9.2.2, Theorem 9.2.5 and Corollary 9.2.6 in [40], we see that the constants $\nu_1, \epsilon_1, d_1, \tilde{d}_1$ depend continuously on the values p and $[w]_{A_p(U)}$, respectively.

From Lemma 3.3.6 and Remark 3.3.7, we have the following lemma.

Lemma 3.3.8. *Let $1 < \gamma_1 \leq \gamma_2 < \infty$ and $A_0 \geq 1$.*

(1) *There exist positive constants ν_0 and $\tilde{d}_0$ depending only on n, γ_1, γ_2 and A_0 such that for any $p \in [\gamma_1, \gamma_2]$ and any weight $w \in A_p(U)$ with $[w]_{A_p(U)} \leq A_0$, we have*

$$\frac{1}{A_0}\left(\frac{|E|}{|C|}\right)^{\gamma_2} \leq \frac{w(E)}{w(C)} \leq d_0\left(\frac{|E|}{|C|}\right)^{\nu_0}$$

for all parabolic cubes $C \subset U$ and all measurable subsets E of C.

(2) *There exist $\epsilon_0 \in (0, \gamma_1 - 1)$ and $\tilde{d}_0 \geq 1$ depending only on n, γ_1, γ_2 and A_0 such that for any $p \in [\gamma_1, \gamma_2]$ and any weight $w \in A_p(U)$ with $[w]_{A_p(U)} \leq A_0$, we have $w \in A_{p-\epsilon_0}(U)$ with $[w]_{A_{p-\epsilon_0}(U)} \leq \tilde{d}_0$.*

We recall basic properties for weighted variable exponent Lebesgue spaces. The results in the following lemma can be found in [28, Chapter 2], by letting $\varphi(x,t) = t^{p(x)}w(x)$.

Lemma 3.3.9. *Let $p(\cdot) : \mathbb{R}^{n+1} \to (1, \infty)$ satisfy* (3.3.1) *and w be a weight.*

(1) *Norm-modular unit ball property:*

$$\|g\|_{L^{p(\cdot)}(U,w)} \leq 1 \quad \Longleftrightarrow \quad \varrho_{p(\cdot),w}(g) \leq 1. \tag{3.3.10}$$

(2) *Relationship between norm and modular:*

$$\begin{aligned}&\min\left\{\left(\varrho_{p(\cdot),w}(g)\right)^{\frac{1}{\gamma_1}}, \left(\varrho_{p(\cdot),w}(g)\right)^{\frac{1}{\gamma_2}}\right\} \\ &\leq \|g\|_{L^{p(\cdot)}(U,w)} \leq \max\left\{\left(\varrho_{p(\cdot),w}(g)\right)^{\frac{1}{\gamma_1}}, \left(\varrho_{p(\cdot),w}(g)\right)^{\frac{1}{\gamma_2}}\right\}.\end{aligned} \tag{3.3.11}$$

(3) *Hölder's inequality: For* $q(\cdot):\mathbb{R}^{n+1}\to(1,\infty)$, *let* $\frac{1}{s(\cdot)} := \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$. *Then we have*

$$\|fg\|_{L^{s(\cdot)}(U,w)} \le 2\|f\|_{L^{p(\cdot)}(U,w)}\|g\|_{L^{q(\cdot)}(U,w)} \tag{3.3.12}$$

(4) $C_0^\infty(U)$ *is dense in* $L^{p(\cdot)}(U,w)$.

(5) $L^{p'(\cdot)}(U,w^{-1/(p(\cdot)-1)})$ *is isomorphic to the dual space* $(L^{p(\cdot)}(U,w))^*$ *of the space* $L^{p(\cdot)}(U,w)$ *in the sense that for* $g\in L^{p'(\cdot)}(U,w^{-1/(p(\cdot)-1)})$, *we define* $J_g\in(L^{p(\cdot)}(U,w))^*$ *by*

$$J_g(f) := \int_U fg\,dz.$$

In particular, there exists $c=c(\gamma_1,\gamma_2,w)\ge 1$ *such that*

$$\frac{1}{c}\|g\|_{L^{p'(\cdot)}(U,w^{-1/(p(\cdot)-1)})} \le \|J_g\|_{(L^{p(\cdot)}(U,w))^*} \le c\|g\|_{L^{p'(\cdot)}(U,w^{-1/(p(\cdot)-1)})}.$$

We next show two properties of $A_p(\cdot)$ class. The first one is duality and the second one is monotonicity. Similar results can be found in [27, 29]. In contrast with [27, 29], however, we adopted parabolic cubes in the definition of $A_{p(\cdot)}$ class, Definition 3.3.2.

Lemma 3.3.10. *Let* $p(\cdot)\in\mathcal{P}^{\log}_{\pm}$ *and* $U\subset\mathbb{R}^{n+1}$ *be bounded. Then we have the relation that*

$$w\in A_{p(\cdot)}(U) \quad\Longleftrightarrow\quad w^{-1/(p(\cdot)-1)}\in A_{p'(\cdot)}(U).$$

Proof. It suffices to show that $w\in A_{p(\cdot)}(U)$ implies $w^{-1/(p(\cdot)-1)}\in A_{p'(\cdot)}(U)$, since this means that this reverse is also valid. Suppose $w\in A_{p(\cdot)}(U)$. Then we will show that

$$\begin{aligned}&[w^{-1/(p(\cdot)-1)}]_{A_{p'(\cdot)}(U)} \\ &= \sup_{C\subset U}|C|^{-(p')_C}\|w^{-1/(p(\cdot)-1)}\|_{L^1(C)}\|w^{1/(p(\cdot)-1)}\|_{L^{p(\cdot)/p'(\cdot)}(C)}<\infty.\end{aligned} \tag{3.3.13}$$

Here, $(p')_C$ is the harmonic mean of p' in C and $(p')_C=(p_C)'$. We first compute

$$|C|^{-(p')_C} = |C|^{-p_C/(p_C-1)} = |C|^{-p_C/(p^+_C-1)}|C|^{-p_C(p^+_C-p_C)/\{(p^+_C-1)(p_C-1)\}}.$$

If $|C| \geq 1$, we have

$$|C|^{-(p')_C} \leq |C|^{-p_C/(p_C^+-1)},$$

and if $|C| = |C_r(\xi)| = (2r)^{n+2} \leq 1$, we have from (3.3.4) that

$$|C|^{-p_C(p_C^+-p_C)/\{(p_C^+-1)(p_C-1)\}} \leq \left(\frac{1}{2r}\right)^{\theta(2\sqrt{2}r)(n+2)\gamma_2/(\gamma_1-1)^2} \leq c$$

and so

$$|C|^{-(p')_C} \leq c|C|^{-p_C/(p_C^+-1)}.$$

As for $\|w^{-1/(p(\cdot)-1)}\|_{L^1(C)}$ and $\|w^{1/(p(\cdot)-1)}\|_{L^{p(\cdot)/p'(\cdot)}(C)}$, we estimate by (3.3.11) that

$$\begin{aligned}\|w^{-1/(p(\cdot)-1)}\|_{L^1(C)} &= \left(\int_C w^{-1/(p(z)-1)}\,dz\right)^{(p_C^+-1)/(p_C^+-1)} \\ &\leq \max\left\{w(U)^{(\gamma_2-\gamma_1)(\gamma_1-1)}, 1\right\} \|w^{-1}\|^{1/(p_C^+-1)}_{L^{p'(\cdot)/p(\cdot)}(C)}\end{aligned}$$

and

$$\|w^{1/(p(\cdot)-1)}\|_{L^{p(\cdot)/p'(\cdot)}(C)} \leq \max\left\{w(U)^{(\gamma_2-\gamma_1)/(\gamma_1-1)^2}, 1\right\} \|w\|^{1/(p_C^+-1)}_{L^1(C)}.$$

Therefore, we have

$$\begin{aligned}&|C|^{-(p')_C}\|w^{-1/(p(\cdot)-1)}\|_{L^1(C)}\|w^{1/(p(\cdot)-1)}\|_{L^{p(\cdot)/p'(\cdot)}(C)} \\ &\quad\leq c[w]^{1/(p_C^+-1)}_{A_{p(\cdot)}} \leq c[w]^{1/(\gamma_1-1)}_{A_{p(\cdot)}},\end{aligned}$$

which implies (3.3.13). □

Lemma 3.3.11. *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}_{\pm}$ with $1 < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty$ and $1 < \gamma_3 \leq p(\cdot) \leq \gamma_4 < \infty$. If $p(\cdot) \leq q(\cdot)$ and $U \subset \mathbb{R}^{n+1}$ be bounded, then there exists $c_m \geq 1$ depending only on $n, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ and the log-Hölder constants of $p(\cdot)$ and $q(\cdot)$ such that*

$$[w]_{A_{q(\cdot)}(U)} \leq c_m \max\left\{|U|^{\gamma_2-\gamma_1+\gamma_4-\gamma_3}, 1\right\}[w]_{A_{p(\cdot)}(U)}.$$

In particular, if $q(\cdot)$ is a constant function, then the constant c_m depends only on n, γ_1, γ_2 and the log-Hölder constant of $p(\cdot)$.

Proof. For a parabolic cube $C = C_r(\xi) \subset U$, we first observe from (3.3.11) and (3.3.12) that

$$\begin{aligned}\|w^{-1}\|_{L^{q'(\cdot)/q(\cdot)}(C)} &\leq 2\|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)}\|1\|_{L^{s(\cdot)}(C)} \\ &\leq 2\|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)} \max\left\{|C|^{1/s_C^+}, |C|^{1/s_C^-}\right\},\end{aligned}$$

where $s(\cdot)^{-1} = q(\cdot) - p(\cdot)$. Note that if $|C| \geq 1$ we see

$$\max\left\{|C|^{1/s_C^+}, |C|^{1/s_C^-}\right\}|C|^{p_C - q_C} \leq |C|^{p_C^+ - p_C^- + q_C^+ - q_C^-} \leq |U|^{\gamma_2-\gamma_1+\gamma_2-\gamma_1},$$

and if $|C| = (2r)^{n+2} \leq 1$ we see

$$\begin{aligned}\max\left\{|C|^{1/s_C^+}, |C|^{1/s_C^-}\right\}|C|^{p_C - q_C} &\leq |C|^{-(p_C^+ - p_C^-)-(q_C^+ - q_C^-)} \\ &\leq \left(\frac{1}{2r}\right)^{(\theta_p(2\sqrt{2}r)+\theta_q(2\sqrt{2}r))(n+2)} \leq c,\end{aligned}$$

where θ_p and θ_q are the modulus of constant of $p(\cdot)$ and $q(\cdot)$, respectively. Therefore, we have

$$\begin{aligned}&|C|^{-q_C}\|w\|_{L^1(C)}\|w^{-1}\|_{L^{q'(\cdot)/q(\cdot)}(C)} \\ &\quad\leq \max\left\{|C|^{1/s_C^+}, |C|^{1/s_C^-}\right\}|C|^{p_C-q_C}|C|^{-p_C}\|w\|_{L^1(C)}\|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)} \\ &\quad\leq c\max\left\{|U|^{\gamma_2-\gamma_1+\gamma_4-\gamma_3}, 1\right\}|C|^{-p_C}\|w\|_{L^1(C)}\|w^{-1}\|_{L^{p'(\cdot)/p(\cdot)}(C)}.\end{aligned}$$

This implies the desired result. □

The following lemma plays a crucial role in Chapter 3.3.3 and Chapter 3.3.4.

Lemma 3.3.12. *Let* $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$, $w \in A_{p(\cdot)}$ *and* $U \subset \mathbb{R}^{n+1}$ *be bounded. There exist* $\tilde{\gamma}_0 = \tilde{\gamma}_0(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) \in (1, \gamma_1)$ *and* $c = c(n, \gamma_1, \gamma_2, c_{LH}, w, U) > 0$ *such that*

$$\|f\|_{L^{\tilde{\gamma}_0}(U)} \leq c\|f\|_{L^{p(\cdot)}(U,w)}. \tag{3.3.14}$$

Proof. We extend f from U to $\mathbb{R}^{n+1}$ by zero. Let $C = C_r(\xi)$ be a parabolic cube. We first note that if $|C| \leq 1$ we have from Lemma 3.3.11 that $w \in A_q(C)$ with $[w]_q(U) \leq c_m[w]_{p(\cdot)}$ for all $q \geq p(\cdot)$ in C and for some $c_m = c_m(n, \gamma_1, \gamma_2, c_{LH})$. Therefore, since $p_C^+ \geq p(\cdot)$ in C and $\gamma_1 \leq p_C^+ \leq \gamma_2$, applying (2) of Lemma 3.3.8 to $A_0 = c_m[w]_{A_{p(\cdot)}}$, there exists $\epsilon_0 = \epsilon_0(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) \in (0, \gamma_1 - 1)$ such that $w \in A_{p_C^+ - \epsilon_0}(C)$ for all parabolic cubes C with $|C| \leq 1$, where $[w]_{A_{p_C^+-\epsilon_0}(C)}$ depends only on n, γ_1, γ_2, c_{LH} and $[w]_{A_{p(\cdot)}}$.

We now consider parabolic cubes C such that

$$|C| = (2r)^{n+1} \leq 1 \quad \text{and} \quad \theta(2\sqrt{2}r) \leq \frac{\epsilon_0}{4}. \tag{3.3.15}$$

Note that $p_C^+ - \epsilon_0/2 \leq p(\cdot)$ in C. Moreover, we infer from (3.3.11) and (3.3.12) that

$$\begin{aligned}
\|f\|_{L^{p_C^+-\epsilon_0/2}(C,w)} &\leq 2\|1\|_{L^{(1/(p_C^+-\epsilon_0/2)-1/p(\cdot))^{-1}}(C,w)} \|f\|_{L^{p(\cdot)}(C,w)} \\
&\leq 2\max\left\{ w(C)^{\frac{1}{p_C^+-\epsilon_0/2}-\frac{1}{p_C^+}}, w(C)^{\frac{1}{p_C^+-\epsilon_0/2}-\frac{1}{p_C^-}} \right\} \|f\|_{L^{p(\cdot)}(C,w)} \\
&\leq 2\max\left\{ w(C)^{-\frac{1}{p_C^+}}, w(C)^{-\frac{1}{p_C^-}} \right\} w(C)^{\frac{1}{p_C^+-\epsilon_0/2}} \|f\|_{L^{p(\cdot)}(C,w)} \\
&\leq 2\max\left\{ w(C)^{-\frac{1}{\gamma_1}}, 1 \right\} w(C)^{\frac{1}{p_C^+-\epsilon_0/2}} \|f\|_{L^{p(\cdot)}(C,w)}
\end{aligned}$$

and from (3.3.9) that

$$\left(\fint_C |f|^{\frac{p_C^+-\epsilon_0/2}{p_C^+-\epsilon_0}}\, dz \right)^{p_C^+-\epsilon_0} \leq \frac{[w]_{Ap_C^+-\epsilon_0(C)}}{w(C)} \int_C |f|^{p_C^+-\epsilon_0/2} w(z)\, dz.$$

Consequently, letting

$$\tilde{\gamma}_0 := \frac{\gamma_2 - \epsilon_0/2}{\gamma_2 - \epsilon_0} = 1 + \frac{\epsilon_0}{2(\gamma_2 - \epsilon_0)} \tag{3.3.16}$$

we have

$$\begin{aligned}
\left(\fint_C |f|^{\tilde{\gamma}_0}\, dz \right)^{\frac{1}{\tilde{\gamma}_0}} &\leq \left(\fint_C |f|^{\frac{p_C^+-\epsilon_0/2}{p_C^+-\epsilon_0}}\, dz \right)^{\frac{p_C^+-\epsilon_0}{p_C^+-\epsilon_0/2}} \\
&\leq 2[w]_{Ap_C^+-\epsilon_0(C)}^{\frac{1}{p_C^+-\epsilon_0/2}} \max\left\{ w(C)^{-\frac{1}{\gamma_1}}, 1 \right\} \|f\|_{L^{p(\cdot)}(C,w)},
\end{aligned}$$

and hence,

$$\|f\|_{L^{\tilde{\gamma}_0}(C)} \leq 2[w]_{A_{p_C^+ - \epsilon_0}(C)}^{\frac{1}{p_C^+ - \epsilon_0/2}} \max\left\{ w(C)^{-\frac{1}{\gamma_1}}, 1 \right\} \|f\|_{L^{p(\cdot)}(C,w)}, \qquad (3.3.17)$$

for all C satisfying (3.3.15). By a standard covering argument, the desired estimate (3.3.14) follows from the previous estimate (3.3.17).

□

3.3.3 Interior and boundary weighted $W^{2,1}_{p(\cdot)}$-estimates

In this section, we establish interior and boundary *a priori* weighted $W^{2,1}_{p(\cdot)}$-estimates, which are a core part of the proof of our main result, Theorem 3.3.4.

The following is the main theorem in this section.

Theorem 3.3.13. *Let $p(\cdot) \in \mathcal{P}^{\log}_{\pm}$ with (3.3.1), the log-Hölder constant $c_{LH} > 0$ and the modulus of continuity $\theta(\cdot)$, and suppose $w \in A_{p(\cdot)}$. Then there exists a small $\rho_0 = \rho_0(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) \in (0,1)$ such that the following hold:*
for any $\rho \in (0, \rho_0]$, there exists a small $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho}))$ $\in (0,1)$ such that

(i) (Interior estimates) if $\mathbf{A}$ is $(\delta, 4\rho)$-vanishing and $f \in L^{p(\cdot)}(Q_{4\rho}, w)$, then for any solution $u \in W^{2,1}_{p(\cdot)}(Q_{4\rho}, w)$ of

$$u_t - a_{ij} D_{ij} u = f \quad \text{in } Q_{4\rho},$$

we have

$$\begin{aligned} &\|u_t\|_{L^{p(\cdot)}(Q_\rho,w)} + \|D^2 u\|_{L^{p(\cdot)}(Q_\rho,w)} \\ &\leq c\rho^{-\frac{(n+2)\gamma_2}{\gamma_1}} \left(\|f\|_{L^{p(\cdot)}(Q_{4\rho},w)} + \frac{1}{\rho^2} \|u\|_{L^{p(\cdot)}(Q_{4\rho},w)} \right) \end{aligned} \qquad (3.3.18)$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$,

(ii) (Boundary estimates) if $\mathbf{A}$ is $(\delta, 4\rho)$-vanishing and $f \in L^{p(\cdot)}(Q^+_{4\rho}, w)$, then for any solution $u \in W^{2,1}_{p(\cdot)}(Q^+_{4\rho}, w)$ of

$$\left\{ \begin{array}{rcll} u_t - a_{ij} D_{ij} u & = & f & \text{in} \quad Q^+_{4\rho}, \\ u & = & 0 & \text{on} \quad T_{4\rho}, \end{array} \right. \qquad (3.3.19)$$

we have

$$\begin{aligned} &\|u_t\|_{L^{p(\cdot)}(Q_\rho^+,w)} + \|D^2u\|_{L^{p(\cdot)}(Q_\rho^+,w)} \\ &\quad \le c\rho^{-\frac{(n+2)\gamma_2}{\gamma_1}} \left(\|f\|_{L^{p(\cdot)}(Q_{4\rho}^+,w)} + \frac{1}{\rho^2}\|u\|_{L^{p(\cdot)}(Q_{4\rho}^+,w)} \right) \end{aligned} \tag{3.3.20}$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$.

Since the proof of the interior estimate (3.3.18) in Theorem 3.3.13 is analogous to that of the boundary estimate (3.3.20) in Theorem 3.3.13, we shall only establish the boundary estimate (3.3.20). We divide the proof of the boundary case into several subsections.

We first take ρ_0 as follows. Recall $\epsilon_0 = \epsilon_0(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) \in (0, \gamma_1 - 1)$ determined in (2) of Lemma 3.3.8 with $A_0 = c_m[w]_{A_{p(\cdot)}}$. Without loss of generality, we assume that

$$\epsilon_0 \le \frac{2\gamma_2}{3}. \tag{3.3.21}$$

Then we take $\rho_0 > 0$ to be the largest number satisfying

$$\rho_0 \le \frac{1}{8}, \quad |C_{4\rho_0}| \le 1 \quad \text{and} \quad \theta(8\sqrt{2}\rho_0) \le \min\left\{\frac{\gamma_1\epsilon_0}{2\gamma_2 - \epsilon_0}, \frac{\epsilon_0}{4}, 1\right\}. \tag{3.3.22}$$

From now on, we fix $\rho \le \rho_0$ and suppose that $\mathbf{A}$ is $(\delta, 4\rho)$-vanishing, where $\delta > 0$ will be determined later; see Remark 3.3.16. Set

$$p^- := \inf_{z\in Q_{2\rho}^+} p(z) \quad \text{and} \quad p^+ := \sup_{z\in Q_{2\rho}^+} p(z)$$

and recalling $\tilde{\gamma}_0 = 1 + \frac{\epsilon_0}{2(\gamma_2-\epsilon_0)} < \gamma_1$ in (3.3.16) of Lemma 3.3.12, define

$$\gamma_0 := \frac{1+\tilde{\gamma}_0}{2} = 1 + \frac{\epsilon_0}{4(\gamma_2 - \epsilon_0)} = \frac{4\gamma_2 - 3\epsilon_0}{4(\gamma_2 - \epsilon_0)}. \tag{3.3.23}$$

Then we see that

$$1 < \gamma_0 < \tilde{\gamma}_0 < \gamma_1 \le p^- \le p^+ \le \gamma_2 < +\infty.$$

Moreover, using the restriction $\theta(4\rho) \le \theta(4\rho_0) \le \min\left\{\frac{\gamma_1\epsilon_0}{2\gamma_2-\epsilon_0}, \frac{\epsilon_0}{4}, 1\right\}$ as in

(3.3.22), along with (3.3.21), we obtain that

$$\frac{\gamma_0 p(z)}{p^-} \le \gamma_0 \left(1 + \frac{\theta(4\rho)}{\gamma_1}\right) \le \gamma_0 \left(1 + \frac{\epsilon_0}{2\gamma_2 - \epsilon_0}\right) = \tilde{\gamma}_0 \quad \text{for } z \in Q^+_{2\rho}, \tag{3.3.24}$$

and

$$\begin{aligned} p^+ - \epsilon_0 &= p^+ - \frac{4(\gamma_2 - \epsilon_0)p^-}{4\gamma_2 - 3\epsilon_0} + \frac{p^-}{\gamma_0} - \epsilon_0 \\ &= p^+ - p^- + \frac{\epsilon_0 p^-}{4\gamma_2 - 3\epsilon_0} + \frac{p^-}{\gamma_0} - \epsilon_0 \\ &\le \theta(4\rho) + \frac{\epsilon_0\gamma_2}{4\gamma_2 - 3\epsilon_0} + \frac{p^-}{\gamma_0} - \epsilon_0 \le \frac{\epsilon_0}{4} + \frac{\epsilon_0}{2} + \frac{p^-}{\gamma_0} - \epsilon_0 \\ &< \frac{p^-}{\gamma_0}. \end{aligned} \tag{3.3.25}$$

To simplify the proof of (3.3.20), we assume

$$\|f\|_{L^{p(\cdot)}(Q^+_{4\rho},w)} \le 1 \quad \text{and} \quad \|u\|_{L^{p(\cdot)}(Q^+_{4\rho},w)} \le \rho^2, \tag{3.3.26}$$

and then show

$$\|u_t\|_{L^{p(\cdot)}(Q^+_{\rho},w)} + \|D^2 u\|_{L^{p(\cdot)}(Q^+_{\rho},w)} \le c\rho^{-\frac{(n+2)\gamma_2}{\gamma_1}} \tag{3.3.27}$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$. In fact, by virtue of the standard normalization argument, defining

$$\tilde{u} := \frac{u}{\|f\|_{L^{p(\cdot)}(Q^+_{4\rho},w)} + \frac{1}{\rho^2}\|u\|_{L^{p(\cdot)}(Q^+_{4\rho},w)}}$$

and

$$\tilde{f} := \frac{f}{\|f\|_{L^{p(\cdot)}(Q^+_{4\rho},w)} + \frac{1}{\rho^2}\|u\|_{L^{p(\cdot)}(Q^+_{4\rho},w)}}$$

for u and f given in Theorem 3.3.13 (ii), we have that

$$\|\tilde{f}\|_{L^{p(\cdot)}(Q^+_{4\rho},w)} \le 1, \quad \|\tilde{u}\|_{L^{p(\cdot)}(Q^+_{4\rho},w)} \le \rho^2$$

and $\tilde{u}$ is a solution of

$$\begin{cases} \tilde{u}_t - a_{ij}D_{ij}\tilde{u} &= \tilde{f} \quad \text{in} \quad Q^+_{4\rho}, \\ \tilde{u} &= 0 \quad \text{on} \quad T_{4\rho}. \end{cases}$$

Then (3.3.27) implies

$$\|\tilde{u}_t\|_{L^{p(\cdot)}(Q_\rho^+,w)} + \|D^2\tilde{u}\|_{L^{p(\cdot)}(Q_\rho^+,w)} \leq c\rho^{-\frac{(n+2)\gamma_2}{\gamma_1}},$$

which means the desired estimate (3.3.20). Therefore, from now on, we prove the estimate (3.3.27), instead of (3.3.20), under the additional assumption (3.3.26).

We remark that in view of Lemma 3.3.12, especially (3.3.17), we have from (3.3.26) and the restriction $|C_{4\rho}| \leq |C_{4\rho_0}| \leq 1$ and $\theta(8\sqrt{2}\rho) \leq \theta(8\sqrt{2}\rho_0) \leq \frac{\epsilon_0}{4}$ in (3.3.22) that

$$\|f\|_{L^{\tilde{\gamma}_0}(Q_{4\rho}^+)} \leq c \quad \text{and} \quad \|u\|_{L^{\tilde{\gamma}_0}(Q_{4\rho}^+)} \leq c\rho^2 \tag{3.3.28}$$

for some $c = c(n, \gamma_1, \gamma_2, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 0$. Therefore, recalling (ii) of Lemma 3.1.1 we see

$$\|u_t\|_{L^{\tilde{\gamma}_0}(Q_{2\rho}^+)} + \|D^2u\|_{L^{\tilde{\gamma}_0}(Q_{2\rho}^+)} \leq c$$

and hence, it follows from (3.3.11) that

$$\int_{Q_{2\rho}^+} |u_t|^{\tilde{\gamma}_0}\, dz + \int_{Q_{2\rho}^+} |D^2u|^{\tilde{\gamma}_0}\, dz \leq c \tag{3.3.29}$$

for some $c = c(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, \theta(\cdot), [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 0$.

Hereafter, in this section, we denote by the letter c any positive constant depending only on $n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}$ and $w(Q_{4\rho})$, and it is possibly varying from line to line.

Let us define

$$\lambda_0 := \fint_{Q_{2\rho}^+} \left[|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}\left(|f|^{\frac{\gamma_0 p(z)}{p^-}} + 1\right)\right] dz > 1. \tag{3.3.30}$$

We choose any s_1, s_2 with $1 \leq s_1 < s_2 \leq 2$, and for $\lambda > 0$, define the upper-level set

$$E(\lambda) := \left\{z \in Q_{s_1\rho}^+ : |u_t(z)|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u(z)|^{\frac{\gamma_0 p(z)}{p^-}} > \lambda\right\}. \tag{3.3.31}$$

Using a stopping time argument and the Vitali covering lemma, we will find an appropriate covering of the upper-level set $E(\lambda)$, where λ is large enough so that

$$\lambda \geq A\lambda_0, \text{ where } A := \left(\frac{240}{s_2 - s_1}\right)^{n+2}. \tag{3.3.32}$$

For each $\xi \in E(\lambda)$, define a continuous function $\Phi_\xi : (0, (s_2 - s_1)\rho] \to [0, \infty)$ by

$$\Phi_\xi(\tau) := \fint_{Q_\tau^+(\xi)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz.$$

Note that $Q_\tau^+(\xi) \subset Q_{s_2\rho}^+ \subset Q_{2\rho}^+$ for $\tau \in (0, (s_2 - s_1)\rho)$. Then, for any $\xi \in E(\lambda)$ and any $\tau \in \left[\frac{(s_2-s_1)\rho}{120}, (s_2 - s_1)\rho\right]$, we have

$$\begin{aligned}
\Phi_\xi(\tau) &= \fint_{Q_\tau^+(\xi)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz \\
&\leq \frac{\left|Q_{2\rho}^+\right|}{\left|Q_\tau^+(\xi)\right|} \fint_{Q_{2\rho}^+} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz \\
&= \left(\frac{2\rho}{\tau}\right)^{n+2} \fint_{Q_{2\rho}^+} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz \\
&< \left(\frac{240}{s_2 - s_1}\right)^{n+2} \fint_{Q_{2\rho}^+} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz \\
&< A\lambda_0 \leq \lambda,
\end{aligned}$$

where the inequalities in the last two line come from (3.3.30) and (3.3.32). On the other hand, Lebesgue's differentiation theorem leads us to obtain

$$\lim_{\tau \to 0} \Phi_\xi(\tau) = \lim_{\tau \to 0} \fint_{Q_\tau^+(\xi)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}} \right) dz > \lambda,$$

for almost every $\xi \in E(\lambda)$. Hence, for almost every $\xi \in E(\lambda)$, there exists

$$\tau_\xi \in \left(0, \frac{(s_2 - s_1)\rho}{120}\right)$$

such that

$$\Phi_\xi(\tau_\xi) = \lambda \quad \text{and} \quad \Phi_\xi(\tau) < \lambda, \quad \text{for all } \tau \in (\tau_\xi, (s_2 - s_1)\rho].$$

According to the Vitali covering lemma, we consequently find $\xi^k \in E(\lambda)$

and $\tau_k := \tau_{\xi^k} \in \left(0, \frac{(s_2-s_1)\rho}{120}\right)$, $k = 1, 2, \dots$, such that the family of parabolic cylinders $\left\{Q^+_{\tau_k}(\xi^k)\right\}_{k=1}^{\infty}$ is mutually disjoint and satisfies the relation

$$E(\lambda) \subset \bigcup_{k=1}^{\infty} Q^+_{5\tau_k}(\xi^k) \subset Q^+_{s_2\rho}, \tag{3.3.33}$$

except a Lebesgue measure zero set. Note that for each k we have

$$\Phi_{\xi^k}(\tau_k) = \lambda \quad \text{and} \quad \Phi_{\xi^k}(\tau) < \lambda, \quad \text{for all } \tau \in (\tau_k, (s_2 - s_1)\rho]. \tag{3.3.34}$$

Lemma 3.3.14. *Under the above settings, we have for each $k = 1, 2, \dots,$*

$$\begin{aligned} w(C_{\tau_k}(\xi^k)) \le \frac{2c_a}{\lambda^{p^+-\epsilon_0}} \times & \qquad\qquad (3.3.35)\\ \Bigg[\int_{Q^+_{\tau_k}(\xi^k) \cap \{|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda}{4c_a}\}} & \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w\,dz \\ & + \int_{Q^+_{\tau_k}(\xi^k) \cap \{|f|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda\delta}{4c_a}\}} \left(\delta^{-1}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w\,dz \Bigg], \end{aligned}$$

for some $c_a = c_a(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) > 1$.

Proof. By the first equality of (3.3.34), we have

$$\lambda \le \frac{2^{n+1}}{|C_{\tau_k}(\xi^k)|} \int_{Q^+_{\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} + \frac{1}{\delta}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz.$$

Since $w \in A_{p^+-\epsilon_0}(Q_{4\rho})$, we apply (3.3.9) to obtain

$$\begin{aligned} w(C_{\tau_k}(\xi^k)) \ \le\ \frac{c_a}{\lambda^{p^+-\epsilon_0}} \Bigg[& \int_{Q^+_{\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz \\ & + \int_{Q^+_{\tau_k}(\xi^k)} \left(\delta^{-1}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz \Bigg] \end{aligned} \tag{3.3.36}$$

for some $c_a = c_a(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) > 1$.

Note that

$$\int_{Q^+_{\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz$$

$$
\begin{aligned}
&\le \int_{Q^+_{\tau_k}(\xi^k)\cap\{|u_t|^{\frac{\gamma_0 p(z)}{p^-}}+|D^2u|^{\frac{\gamma_0 p(z)}{p^-}}>\frac{\lambda}{4c_a}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}}+|D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w dz \\
&\quad + \frac{\lambda^{p^+-\epsilon_0}}{4c_a} w(C_{\tau_k}(\xi^k)),
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{Q^+_{\tau_k}(\xi^k)} \left(\delta^{-1}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz \\
&\quad \le \int_{Q^+_{\tau_k}(\xi^k)\cap\{|f|^{\frac{\gamma_0 p(z)}{p^-}}>\frac{\lambda\delta}{4c_a}\}} \left(\delta^{-1}|f|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz \\
&\qquad + \frac{\lambda^{p^+-\epsilon_0}}{4c_a} w(C_{\tau_k}(\xi^k)).
\end{aligned}
$$

Therefore, inserting the above two inequalities into (3.3.36), we conclude the desired estimate (3.3.35). □

Now, we seek comparison estimates on each cylinder $Q_{5\tau_k}(\xi^k)$. We first divide the covers $Q_{5\tau_k}(\xi^k)$, $k=1,2,\dots$, into the two cases that the *interior case*: $B_{20\tau_k}(y^k)\subset B^+_{s_2\rho}$ and the *boundary case*: $B_{20\tau_k}(y^k)\not\subset B^+_{s_2\rho}$, i.e. $B_{20\tau_k}(y^k)\cap\{x\in\mathbb{R}^n : x_n<0\}\neq\emptyset$, where $\xi^k := (y^k,s^k)$. In particular, for the boundary case, we can find a point $\tilde{\xi}^k := (\tilde{y}^k,s^k)$ where $\tilde{y}^k\in B_{s_2\rho}(0)\cap\{x\in\mathbb{R}^n : x_n=0\}$ satisfying $|y^k-\tilde{y}^k|<20\tau_k$.

Lemma 3.3.15. *Under the above settings, the following hold:*

(a) (Interior case) If $B_{20\tau_k}(y^k)\subset B^+_{s_2\rho}$, we have

$$
\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\gamma_0}+|D^2u|^{\gamma_0}\right) dz \le c_0\lambda^{\frac{p^-}{p^+_k}}
$$

and

$$
\fint_{Q_{20\tau_k}(\xi^k)} |f|^{\gamma_0} dz \le c_0\lambda^{\frac{p^-}{p^+_k}}\delta^{\frac{\gamma_1}{\gamma_2}}, \tag{3.3.37}
$$

for some $c_0=c_0(n,\Lambda,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}},w(Q_{4\rho}))>1$. Moreover, for any $\epsilon\in(0,1)$, there exist $\delta=\delta(\epsilon,n,\Lambda,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}})>0$ and

$v_k \in W^{2,1}_{\gamma_0}(Q_{20\tau_k}(\xi^k)) \cap W^{2,1}_{\infty}(Q_{5\tau_k}(\xi^k))$ *such that*

$$\fint_{Q_{5\tau_k}(\xi^k)} \left(|(u-v_k)_t|^{\gamma_0} + |D^2(u-v_k)|^{\gamma_0}\right) dz \leq \epsilon c_0 \lambda^{\frac{p^-}{p_k^+}} \tag{3.3.38}$$

and

$$\|(v_k)_t\|^{\gamma_0}_{L^\infty\left(Q_{5\tau_k}(\xi^k)\right)} + \|D^2 v_k\|^{\gamma_0}_{L^\infty\left(Q_{5\tau_k}(\xi^k)\right)} \leq c_1 \lambda^{\frac{p^-}{p_k^+}}, \tag{3.3.39}$$

for some $c_1 = c_1(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$.

(b) *(Boundary case) If* $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$, *we have*

$$\fint_{Q^+_{100\tau_k}(\tilde{\xi}^k)} \left(|u_t|^{\gamma_0} + |D^2 u|^{\gamma_0}\right) dz \leq c_2 \lambda^{\frac{p^-}{p_k^+}}$$

and

$$\fint_{Q^+_{100\tau_k}(\tilde{\xi}^k)} |f|^{\gamma_0} dz \leq c_2 \lambda^{\frac{p^-}{p_k^+}} \delta^{\frac{\gamma_1}{\gamma_2}}, \tag{3.3.40}$$

for some $c_2 = c_2(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$. *Moreover, for any* $\epsilon \in (0,1)$, *there exist* $\delta = \delta(\epsilon, n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}) > 0$ *and* $v_k \in W^{2,1}_{\gamma_0}(Q^+_{100\tau_k}(\tilde{\xi}^k)) \cap W^{2,1}_{\infty}(Q^+_{25\tau_k}(\tilde{\xi}^k))$ *such that*

$$\fint_{Q^+_{25\tau_k}(\tilde{\xi}^k)} \left(|(u-v_k)_t|^{\gamma_0} + |D^2(u-v_k)|^{\gamma_0}\right) dz \leq \epsilon c_2 \lambda^{\frac{p^-}{p_k^+}} \tag{3.3.41}$$

and

$$\|(v_k)_t\|^{\gamma_0}_{L^\infty\left(Q^+_{25\tau_k}(\tilde{\xi}^k)\right)} + \|D^2 v_k\|^{\gamma_0}_{L^\infty\left(Q^+_{25\tau_k}(\tilde{\xi}^k)\right)} \leq c_3 \lambda^{\frac{p^-}{p_k^+}}, \tag{3.3.42}$$

for some $c_3 = c_3(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, w(Q_{4\rho})) > 1$.

Proof. Let us first consider the interior case (a) $B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$. One can easily see that

$$20\tau_k \leq (s_2 - s_1)\rho \leq \rho_0 \text{ and } B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}.$$

For the sake of simplicity, we write

$$p_k^- := \inf_{z\in Q_{20\tau_k}(\xi^k)} p(z) \quad\text{and}\quad p_k^+ := \sup_{z\in Q_{20\tau_k}(\xi^k)} p(z), \tag{3.3.43}$$

and then it follows from (3.3.6) that

$$p_k^+ - p_k^- \le \theta(40\tau_k). \tag{3.3.44}$$

From (3.3.22) we know $40\tau_k \le 1$, $\theta(40\tau_k) \le 1$ and $|Q_{20\tau_k}| \le 1$. Using these facts, along with (3.3.29) and (3.3.44), we deduce

$$\begin{aligned}
&\left[\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\gamma_0} + |D^2u|^{\gamma_0}\right) dz\right]^{p_k^+-p_k^-} \\
&\le \left[\frac{1}{|Q_{20\tau_k}(\xi^k)|}\int_{Q_{2\rho}^+} \left(|u_t|^{\tilde{\gamma}_0} + |D^2u|^{\tilde{\gamma}_0} + 2\right) dz\right]^{p_k^+-p_k^-} \\
&\le c\left(\frac{1}{|Q_{20\tau_k}(\xi^k)|}\right)^{\theta(40\tau_k)} \le c\left(\frac{1}{40\tau_k}\right)^{(n+2)\theta(40\tau_k)} \le c.
\end{aligned} \tag{3.3.45}$$

where the last inequality comes from (3.3.4). In an analogous way to (3.3.45), we can obtain from (3.3.4), (3.3.28) and (3.3.44) that

$$\left(\fint_{Q_{20\tau_k}(\xi^k)} |f|^{\gamma_0} dz\right)^{p_k^+-p_k^-} \le c.$$

According to Hölder's inequality with facts $\gamma_1 \le p_k^+$ and $p^- \le p_k^-$, we then infer from (3.3.34) and (3.3.45) that

$$\begin{aligned}
\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\gamma_0} + |D^2u|^{\gamma_0}\right) dz &\le c\left[\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\gamma_0} + |D^2u|^{\gamma_0}\right) dz\right]^{\frac{p_k^-}{p_k^+}} \\
&\le c\left[\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p_k^-}{p^-}} + |D^2u|^{\frac{\gamma_0 p_k^-}{p^-}}\right) dz\right]^{\frac{p^-}{p_k^+}} \\
&\le c\left[\fint_{Q_{20\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz + 2\right]^{\frac{p^-}{p_k^+}} \le c\lambda^{\frac{p^-}{p_k^+}},
\end{aligned}$$

and moreover, using the same argument above, we also deduce

$$\begin{aligned}\fint_{Q_{20\tau_k}(\xi^k)} |f|^{\gamma_0} dz &\leq c\left(\fint_{Q_{20\tau_k}(\xi^k)} |f|^{\frac{\gamma_0 p(z)}{p^-}} dz + 1\right)^{\frac{p^-}{p_k^+}} \\ &\leq c(\delta\lambda+1)^{\frac{p^-}{p_k^+}} \leq c\lambda^{\frac{p^-}{p_k^+}} \delta^{\frac{\gamma_1}{\gamma_2}},\end{aligned}$$

where the last inequality comes from the fact $1 < \delta\lambda_0 < \delta\lambda$ which is induced by (3.3.30) and (3.3.32). In turn, the desired estimate (3.3.37) follows.

We now rescale $Q_{20\tau_k}(\xi^k)$ to Q_4 by setting

$$h_k(\tilde{z}) := \frac{u\left(5\tau_k(\tilde{x}-y^k), 25\tau_k^2(\tilde{t}-s^k)\right)}{25\tau_k^2\left(c_0\lambda^{\frac{p^-}{p_k^+}}\right)^{\frac{1}{\gamma_0}}},$$

$$g_k(\tilde{z}) := \frac{f\left(5\tau_k(\tilde{x}-y^k), 25\tau_k^2(\tilde{t}-s^k)\right)}{\left(c_0\lambda^{\frac{p^-}{p_k^+}}\right)^{\frac{1}{\gamma_0}}}$$

and

$$(b_{ij}^k(\tilde{z})) := \mathbf{B}_k(\tilde{z}) := \mathbf{A}\left(5\tau_k(\tilde{x}-y^k), 25\tau_k^2(\tilde{t}-s^k)\right),$$

for $\tilde{z} := (\tilde{x}, \tilde{t}) \in Q_4$. By a straightforward calculation, one can check from (3.0.2), the $(\delta, 4\rho)$-vanishing condition of $\mathbf{A}$ and the above resulting estimates (3.3.37) that $\mathbf{B}_k$ also satisfies (3.0.2) with $\mathbf{A}(z)$ replaced by $\mathbf{B}_k(\tilde{z})$,

$$[\mathbf{B}_k]_4 \leq \delta, \quad \fint_{Q_4} \left(|(h_k)_t|^{\gamma_0} + |D^2 h_k|^{\gamma_0}\right) dz \leq 1 \quad \text{and} \quad \fint_{Q_4} |g_k|^{\gamma_0} dz \leq \delta^{\frac{\gamma_1}{\gamma_2}}.$$

Besides, $h_k \in W^{2,1}_{p(\cdot)}(Q_4, w) \subset W^{2,1}_{\gamma_0}(Q_4)$ is a solution of

$$(h_k)_t - b_{ij}^k D_{ij} h = g_k \quad \text{in } Q_4. \tag{3.3.46}$$

Therefore, applying Lemma 3.1.4 and Corollary 3.1.5 to the equation (3.3.46) with $\mathbf{B}$, q and δ replaced by $\mathbf{B}_k$, γ_0 and $\delta^{\frac{\gamma_1}{\gamma_2}}$, respectively, we obtain that there exist a constant matrix $\tilde{\mathbf{B}}_k = (\tilde{b}_{ij}^k)$ and a solution $\tilde{v}_k \in W^{2,1}_{\gamma_0}(Q_4)$ of

$$(\tilde{v}_k)_t - \tilde{b}_{ij}^k D_{ij} \tilde{v}_k = 0 \quad \text{in } Q_4,$$

satisfying

$$\fint_{Q_1} \left(|(h_k - \tilde{v}_k)_t|^{\gamma_0} + |D^2(h_k - \tilde{v}_k)|^{\gamma_0}\right) dz \leq \epsilon$$

and

$$\fint_{Q_4} \left(|(\tilde{v}_k)_t|^{\gamma_0} + |D^2 \tilde{v}_k|^{\gamma_0}\right) dz \leq 1,$$

by choosing sufficiently small $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, \theta(\cdot)) > 0$. Moreover, we also have

$$\|(\tilde{v}_k)_t\|^{\gamma_0}_{L^\infty(Q_1)} + \|D^2 \tilde{v}_k\|^{\gamma_0}_{L^\infty(Q_1)} \leq c.$$

Therefore, letting

$$v_k(z) := 25\tau_k^2 \left(c_0 \lambda^{\frac{p^-}{p_k^+}}\right)^{\frac{1}{\gamma_0}} \tilde{v}_k \left(y^k + \frac{1}{5\tau_k} x, s^k + \frac{1}{25\tau_k^2} t\right)$$

for all $z := (x,t) \in Q_{20\tau_k}(\xi^k)$, we conclude that v_k is in $W^{2,1}_{\gamma_0}(Q_{20\tau_k}(\xi^k)) \cap W^{2,1}_{\infty}(Q_{5\tau_k}(\xi^k))$ and satisfies the estimates (3.3.38) and (3.3.39).

Next we deal with the boundary case (b) $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$. Note that $|y^k - \tilde{y}^k| < 20\tau_k$. From the fact $120\tau_k \leq (s_2 - s_1)\rho \leq \rho_0$, it is clear that

$$B_{5\tau_k}(y^k) \subset B^+_{25\tau_k}(\tilde{y}^k) \subset B^+_{100\tau_k}(\tilde{y}^k) \subset B^+_{120\tau_k}(y^k) \subset B^+_{s_2\rho}. \tag{3.3.47}$$

We abbreviate

$$p_k^- := \inf_{z \in Q^+_{100\tau_k}(\tilde{\xi}^k)} p(z) \quad \text{and} \quad p_k^+ := \sup_{z \in Q^+_{100\tau_k}(\tilde{\xi}^k)} p(z). \tag{3.3.48}$$

We also get from (3.3.6) that

$$p_k^+ - p_k^- \leq \theta(200\tau_k).$$

We recall (3.3.34) to discover

$$\fint_{Q^+_{120\tau_k}(\xi^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz \leq \lambda \text{ and } \fint_{Q^+_{120\tau_k}(\xi^k)} |f|^{\frac{\gamma_0 p(z)}{p^-}} dz \leq \delta\lambda.$$

By means of (3.3.47), we then obtain

$$\fint_{Q^+_{100\tau_k}(\tilde{\xi}^k)} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz \leq 2^{n+2}\lambda$$

and

$$\fint_{Q^+_{100\tau_k}(\tilde{\xi}^k)} |f|^{\frac{\gamma_0 p(z)}{p^-}}\, dz \le 2^{n+2}\delta\lambda.$$

Using an analogous argument to the above interior case (a) by taking into account (3.3.48), the previous two estimates, Lemma 3.1.6 and Corollary 3.1.7, in place of (3.3.43), (3.3.34), Lemma 3.1.4 and Corollary 3.1.5, respectively, we can derive the desired estimates (3.3.40), and find the desired v_k satisfying (3.3.41) and (3.3.42). □

Proof of (3.3.27). For constants c_1 and c_3 given in Lemma 3.3.15, let us set

$$K := (2^{\gamma_0 - 1} c_4)^{\frac{\gamma_2}{\gamma_1}} \quad \text{where} \quad c_4 := \max\{c_1, c_3\}. \tag{3.3.49}$$

Recalling the upper-level set (3.3.31), an elementary calculus yields

$$\begin{aligned}
&\int_{Q^+_{s_1\rho}} \left(|u_t|^{p(z)} + |D^2 u|^{p(z)}\right) w(z)\, dz \qquad\qquad (3.3.50)\\
&\le c \int_{Q^+_{s_1\rho}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{\frac{p^-}{\gamma_0}} w(z)\, dz\\
&= \frac{c\, p^-}{\gamma_0} K^{\frac{p^-}{\gamma_0}} \int_0^\infty \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\, d\lambda\\
&\le c \left(\int_0^{A\lambda_0} \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\, d\lambda + \int_{A\lambda_0}^\infty \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\, d\lambda\right)\\
&\le c \left((A\lambda_0)^{\frac{p^-}{\gamma_0}} w(Q^+_{s_1\rho}) + \int_{A\lambda_0}^\infty \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\, d\lambda\right) =: c(I_1 + I_2).
\end{aligned}$$

Taking into account the definitions of λ_0, A and K in (3.3.30), (3.3.32) and (3.3.49), we deduce from (3.3.24), (3.3.28), and (3.3.29) that

$$\begin{aligned}
I_1 \le \frac{c\, w(Q^+_{s_1\rho})}{(s_2 - s_1)^{\frac{(n+2)p^-}{\gamma_0}}} \Bigg[&\fint_{Q^+_{2\rho}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2 u|^{\frac{\gamma_0 p(z)}{p^-}}\right) dz\\
&+ \frac{1}{\delta} \fint_{Q^+_{2\rho}} \left(|f|^{\frac{\gamma_0 p(z)}{p^-}} + 1\right) dz \Bigg]^{\frac{p^-}{\gamma_0}}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{c\, w(Q_{4\rho})}{(s_2-s_1)^{\frac{(n+2)\gamma_2}{\gamma_0}} |Q_{2\rho}|^{\frac{p^-}{\gamma_0}}} \left[\int_{Q^+_{2\rho}} \left(|u_t|^{\tilde{\gamma}_0} + |D^2u|^{\tilde{\gamma}_0}\right) dz \right.\\
&\qquad\qquad\qquad \left. + \frac{1}{\delta} \int_{Q^+_{2\rho}} \left(|f|^{\tilde{\gamma}_0} + 1\right) dz \right]^{\frac{p^-}{\gamma_0}}\\
&\le \frac{c\left(1+\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}} |Q_\rho|^{-\gamma_2}}{(s_2-s_1)^{\frac{(n+2)\gamma_2}{\gamma_0}}}. \qquad (3.3.51)
\end{aligned}
$$

Now we compute I_2. We start with estimating $w(E(K\lambda))$ for $\lambda \ge A\lambda_0$. We recall the covering $\left\{Q^+_{5\tau_k}(\xi^k)\right\}_{k=1}^{\infty}$ of $E(\lambda)$ in (3.3.33). Since $K \ge 1$, we see $E(K\lambda) \subset E(\lambda)$, and so it follows that

$$
\begin{aligned}
&w(E(K\lambda)) \qquad (3.3.52)\\
&\quad \le \sum_{k=1}^{\infty} w\left(\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\frac{\gamma_0 p(z)}{p^-}} + \left|D^2u(z)\right|^{\frac{\gamma_0 p(z)}{p^-}} > K\lambda\right\}\right)\\
&\quad \le \sum_{k=1}^{\infty} w\left(\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right).
\end{aligned}
$$

In order to estimate the sum of measures of the level sets on the right-hand side of (3.3.52), we should consider two cases, the interior case $B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$ and the boundary case $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$.

For the interior case $B_{20\tau_k}(y^k) \subset B^+_{s_2\rho}$ which means $Q^+_{5\tau_k}(\xi^k) = Q_{5\tau_k}(\xi^k)$, we infer from (3.3.38), (3.3.39), (3.3.43), (3.3.49) and the elementary inequality $(a+b)^\beta \le 2^{\beta-1}(a^\beta + b^\beta)$ for any $a, b > 0$ and $\beta \ge 1$, that

$$
\begin{aligned}
&\left|\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right|\\
&\le \left|\left\{z \in Q_{5\tau_k}(\xi^k) : |(u-v_k)_t(z)|^{\gamma_0} + \left|D^2(u-v_k)(z)\right|^{\gamma_0} > c_1\lambda^{\frac{p^-}{p^+_k}}\right\}\right|\\
&\quad + \left|\left\{z \in Q_{5\tau_k}(\xi^k) : |(v_k)_t(z)|^{\gamma_0} + \left|D^2v_k(z)\right|^{\gamma_0} > c_1\lambda^{\frac{p^-}{p^+_k}}\right\}\right|\\
&\le \left(c_1\lambda^{\frac{p^-}{p^+_k}}\right)^{-1} \int_{Q_{5\tau_k}(\xi^k)} \left(|(u-v_k)_t|^{\gamma_0} + \left|D^2(u-v_k)\right|^{\gamma_0}\right) dz \le \frac{\epsilon c_0}{c_1}\left|C_{5\tau_k}(\xi^k)\right|.
\end{aligned}
$$

Then (1) of Lemma 3.3.8 allows us to discover that

$$\begin{aligned} & w\left(\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right) \\ & \leq c\epsilon^{\nu_0} w(C_{5\tau_k}(\xi^k)) \leq c\epsilon^{\nu_0} w(C_{\tau_k}(\xi^k)). \end{aligned} \tag{3.3.53}$$

Similarly, for the boundary case $B_{20\tau_k}(y^k) \not\subset B^+_{s_2\rho}$, it follows from (3.3.41), (3.3.42), (3.3.47) and (3.3.48) that

$$\begin{aligned} & \left|\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right| \\ & \leq \left|\left\{z \in Q^+_{25\tau_k}(\tilde{\xi}^k) : |(u-v_k)_t(z)|^{\gamma_0} + \left|D^2(u-v_k)(z)\right|^{\gamma_0} > c_3\lambda^{\frac{p^-}{p_k^+}}\right\}\right| \\ & \quad + \left|\left\{z \in Q^+_{25\tau_k}(\tilde{\xi}^k) : |(v_k)_t(z)|^{\gamma_0} + \left|D^2v_k(z)\right|^{\gamma_0} > c_3\lambda^{\frac{p^-}{p_k^+}}\right\}\right| \\ & \leq \left(c_3\lambda^{\frac{p^-}{p_k^+}}\right)^{-1} \int_{Q^+_{25\tau_k}(\tilde{\xi}^k)} \left(|(u-v)_t|^{\gamma_0} + \left|D^2(u-v)\right|^{\gamma_0}\right) dz \leq \frac{\epsilon c_2}{c_3} \left|C_{25\tau_k}(\tilde{\xi}^k)\right|, \end{aligned}$$

and then we apply (1) of Lemma 3.3.8 to find that

$$\begin{aligned} & w\left(\left\{z \in Q^+_{5\tau_k}(\xi^k) : |u_t(z)|^{\gamma_0} + \left|D^2u(z)\right|^{\gamma_0} > (K\lambda)^{\frac{p^-}{p(z)}}\right\}\right) \\ & \leq c\epsilon^{\nu_0} w(C_{25\tau_k}(\tilde{\xi}^k)) \leq c\epsilon^{\nu_0} w(C_{\tau_k}(\tilde{\xi}^k)). \end{aligned} \tag{3.3.54}$$

Inserting (3.3.53) and (3.3.54) into (3.3.52), we eventually obtain from (3.3.35) that

$$\begin{aligned} & w(E(K\lambda)) \leq c\epsilon^{\nu_0} \sum_{k=1}^{\infty} w(C_{\tau_k}(\xi^k)) \\ & \leq \frac{c\epsilon^{\nu_0}}{\lambda^{p^+-\epsilon_0}} \sum_{k=1}^{\infty} \Bigg[\int_{Q^+_{\tau_k}(\xi^k) \cap \{|f|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda\delta}{4ca}\}} \left(\frac{|f|^{\frac{\gamma_0 p(z)}{p^-}}}{\delta}\right)^{p^+-\epsilon_0} w(z)\, dz \\ & \quad + \int_{Q^+_{\tau_k}(\xi^k) \cap \{|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}} > \frac{\lambda}{4ca}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}} + |D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\, dz \Bigg] \end{aligned}$$

$$
\begin{aligned}
&\le \frac{c\epsilon^{\nu_0}}{\lambda^{p^+-\epsilon_0}} \times \\
&\left[\int_{Q^+_{S_2\rho}\cap\{|u_t|^{\frac{\gamma_0 p(z)}{p^-}}+|D^2u|^{\frac{\gamma_0 p(z)}{p^-}}>\frac{\lambda}{4c_a}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}}+|D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz\right. \\
&\left.+\int_{Q^+_{s_2\rho}\cap\{|f|^{\frac{\gamma_0 p(z)}{p^-}}>\frac{\lambda\delta}{4c_a}\}} \left(\frac{|f|^{\frac{\gamma_0 p(z)}{p^-}}}{\delta}\right)^{p^+-\epsilon_0} w(z)\,dz\right].
\end{aligned}
\tag{3.3.55}
$$

Accordingly, this estimate (3.3.55) leads us to discover

$$
\begin{aligned}
I_2 &= \int_{A\lambda_0}^{\infty} \lambda^{\frac{p^-}{\gamma_0}-1} w(E(K\lambda))\,d\lambda \\
&\le c\epsilon^{\nu_0}\int_0^{\infty} \lambda^{\frac{p^-}{\gamma_0}-(p^+-\epsilon_0)-1} \times \\
&\left[\int_{Q^+_{S_2\rho}\cap\{|u_t|^{\frac{\gamma_0 p(z)}{p^-}}+|D^2u|^{\frac{\gamma_0 p(z)}{p^-}}>\frac{\lambda}{4c_a}\}} \left(|u_t|^{\frac{\gamma_0 p(z)}{p^-}}+|D^2u|^{\frac{\gamma_0 p(z)}{p^-}}\right)^{p^+-\epsilon_0} w(z)\,dz\right] d\lambda \\
&+c\epsilon^{\nu_0}\int_0^{\infty} \lambda^{\frac{p^-}{\gamma_0}-(p^+-\epsilon_0)-1}\int_{Q^+_{s_2\rho}\cap\{|f|^{\frac{\gamma_0 p(z)}{p^-}}>\frac{\lambda\delta}{4c_a}\}} \left(\frac{|f|^{\frac{\gamma_0 p(z)}{p^-}}}{\delta}\right)^{p^+-\epsilon_0} w(z)\,dzd\lambda.
\end{aligned}
$$

Then applying the basic identity

$$
\int_U |g(z)|^q\, w(z)\,dz = (q-\tilde{q})\int_0^{\infty}\lambda^{q-\tilde{q}-1}\int_{\{z\in U:|g(x)|>\lambda\}} |g(z)|^{\tilde{q}}\, w(z)\,dzd\lambda
$$

for $q>\tilde{q}\ge 1$, together with (3.3.25) and the additional assumption (3.3.26), we deduce

$$
\begin{aligned}
I_2 &\le c\epsilon^{\nu_0}\left\{\int_{Q^+_{s_2\rho}} \left(|u_t|^{p(z)}+|D^2u|^{p(z)}\right) w(z)\,dz\right. \\
&\qquad \left.+\left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}}\int_{Q^+_{s_2\rho}} |f|^{p(z)} w(z)\,dz\right\} \\
&\le c\epsilon^{\nu_0}\int_{Q^+_{s_2\rho}} \left(|u_t|^{p(z)}+|D^2u|^{p(z)}\right) w(z)dz + c\epsilon\left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}}.
\end{aligned}
\tag{3.3.56}
$$

Therefore, combining (3.3.50), (3.3.51) and (3.3.56), we arrive at

$$\begin{aligned}
&\int_{Q^+_{s_1\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz \\
&\qquad \le c_5\epsilon^{\nu_0} \int_{Q^+_{s_2\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz \\
&\qquad\quad + \frac{c\left(1+\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}} |Q_\rho|^{-\gamma_2}}{(s_2-s_1)^{\frac{(n+2)\gamma_2}{\gamma_0}}} + c\epsilon\left(\frac{1}{\delta}\right)^{\frac{\gamma_2}{\gamma_0}},
\end{aligned}$$

for some $c_5 = c_5(n,\Lambda,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}},w(Q_{4\rho})) > 0$. At this stage, we take $\epsilon = \epsilon(n,\Lambda,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}},w(4\rho)) > 0$ small enough so that

$$0 < c_5\epsilon^{\nu_0} \le \frac{1}{2} \tag{3.3.57}$$

to establish

$$\begin{aligned}
&\int_{Q^+_{s_1\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz \\
&\qquad \le \frac{1}{2}\int_{Q^+_{s_2\rho}} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz + \frac{c\,|Q_\rho|^{-\gamma_2}}{(s_2-s_1)^{\frac{(n+2)\gamma_2}{\gamma_0}}} + c.
\end{aligned}$$

Since s_1 and s_2 with $1 \le s_1 < s_2 \le 2$ are arbitrary, we apply the standard iteration lemma [41, Lemma 4.3] to conclude that

$$\int_{Q^+_\rho} \left(|u_t|^{p(z)} + |D^2u|^{p(z)}\right) w(z)\,dz \le c\,|Q_\rho|^{-\gamma_2} + c \le c_6\rho^{-(n+2)\gamma_2} \tag{3.3.58}$$

for some $c_6 = c_6(n,\Lambda,\gamma_1,\gamma_2,c_{LH},[w]_{A_{p(\cdot)}},w(Q_{4\rho})) > 1$. By virtue of (3.3.11), we consequently obtain the desired estimate (3.3.27). This completes the proof. □

Remark 3.3.16. From the choice of $\epsilon > 0$ in (3.3.57), one can select $\delta > 0$ depending only on $n,\Lambda,\gamma_1,\gamma_2,[w]_{A_{p(\cdot)}}$ and $w(Q_{4\rho})$.

3.3.4 Global weighted $W^{2,1}_{p(\cdot)}$-estimates

The proof of our main result, Theorem 3.3.4, proceeds in three steps. In the first step we show that it suffices to derive the estimate (3.3.7) only for the solutions u of (3.0.1) belonging to $W^{2,1}_{p(\cdot)}(\Omega_T,w)$. Then in the next two steps, by using standard covering and flattening arguments, we obtain the *a priori* estimate (3.3.7) from the interior and boundary *a priori* weighted estimates that have been established in the previous section. In what follows, we denote by c a universal constant being dependent only on $n, \Lambda, \gamma_1, \gamma_2, \theta(\cdot), w, \Omega$ and R, and possibly varying from line to line.

Proof. **Step1. Approximation:** We first suppose that we have the *a priori* estimate, that is, the estimate (3.3.7) holds for any $W^{2,1}_{p(\cdot)}(\Omega_T, w)$-solution of the problem (3.0.1). To get rid of this *a priori* assumption, we show that the solution u of the problem (3.0.1) can be suitably approximated by solutions u_k, $k = 1, 2, \dots$, in $W^{2,1}_{p(\cdot)}(\Omega_T, w)$ to regular equations.

Given $\mathbf{A} = (a_{ij})$, we choose a sequence $\{\mathbf{A}^k\}_{k=1}^{\infty} = \{(a^k_{ij})\}_{k=1}^{\infty}$ of smooth matrix functions satisfying the uniform parabolicity condition with the constant Λ and (δ, R)-vanishing property, which converges to $\mathbf{A} = (a_{ij})$ in $L^{\alpha}(\Omega_T)$ for each $1 < \alpha < \infty$. For instance we may define $(a^k_{ij}) := (a_{ij} * \varphi_{1/k})$, where $\varphi_{1/k}(x) := k^n \varphi(kx)$ and φ is a standard mollification function. On the other hand, for given $f \in L^{p(\cdot)}(\Omega_T, w)$, we also find a sequence $\{f_k\}_{k=1}^{\infty}$ of smooth functions in $C_0^{\infty}(\Omega_T)$ converging to f in $L^{p(\cdot)}(\Omega_T, w)$ and satisfying that

$$\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} \leq \|f\|_{L^{p(\cdot)}(\Omega_T,w)} + 1 \quad \text{for all } k = 1, 2, \dots. \tag{3.3.59}$$

Since $w \in A_{p(\cdot)} \subset A_{\gamma_2+1}$ and $w^{-1/(p(\cdot)-1)} \in A_{p'(\cdot)} \subset A_{\gamma_1/(\gamma_1-1)+1}$, by Lemma 3.3.10 and Lemma 3.3.11, we note that in view of (1) of Lemma 3.3.6, there exist positive constants ν_1 and $\tilde{\nu}_1$ such that $w \in L^{1+\nu_1}(\mathbb{R}^{n+1})$ and $w^{-1/(p(\cdot)-1)} \in L^{1+\tilde{\nu}_1}(\mathbb{R}^{n+1})$. Therefore, we have that for $g \in L^{\frac{\gamma_2(1+\nu_1)}{\nu_1}}(\Omega_T)$,

$$\int_{\Omega_T} |g|^{\gamma_2+1} w \, dz \leq \left(\int_{\Omega_T} |g|^{\frac{(\gamma_2+1)(1+\nu_1)}{\nu_1}} dz \right)^{\frac{\nu_1}{1+\nu_1}} \left(\int_{\Omega_T} w^{1+\nu_1} \, dz \right)^{\frac{1}{1+\nu_1}},$$

from which and (3.3.12) one can find $q_1 = \frac{(\gamma_2+1)(1+\nu_1)}{\nu_1} \in (\gamma_2 + 1, \infty)$ such that

$$L^{q_1}(\Omega_T) \hookrightarrow L^{\gamma_2+1}(\Omega_T, w) \hookrightarrow L^{p(\cdot)}(\Omega_T, w). \tag{3.3.60}$$

In the same argument, there exists $q_2 \in (\gamma_1/(\gamma_1 - 1) + 1, \infty)$ such that

$$L^{q_2}(\Omega_T) \hookrightarrow L^{\gamma_1/(\gamma_1-1)+1}(\Omega_T, w^{-1/(p(\cdot)-1)}) \hookrightarrow L^{p'(\cdot)}(\Omega_T, w^{-1/(p(\cdot)-1)}). \tag{3.3.61}$$

Since $\mathbf{A}^k$ and f_k are smooth, according to [9, Theorem 4.3], there exists the unique solution $u_k \in W^{2,1}_{q_1}(\Omega_T)$ of

$$\begin{cases} (u_k)_t - a^k_{ij} D_{ij} u_k = f_k & \text{in} \quad \Omega_T, \\ u_k = 0 & \text{on} \quad \partial\Omega_T. \end{cases} \tag{3.3.62}$$

We then see from (3.3.60) that $u_k \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$. Hence, by the *a priori* assumption we have the estimate

$$\|u_k\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \leq c\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)}.$$

Moreover, it follows from (3.3.59) that

$$\|u_k\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \leq c\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} \leq c\left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + 1\right), \tag{3.3.63}$$

where c is independent of k, and so $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{2,1}_{p(\cdot)}(\Omega_T, w)$. Therefore, there exist a subsequence, which is still denoted by $\{u_k\}_{k=1}^{\infty}$, and a function $u_0 \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$ such that

$$u_k \rightharpoonup u_0 \;\text{ weakly in } W^{2,1}_{p(\cdot)}(\Omega_T, w).$$

On the other hand, for the sequence $\{\mathbf{A}^k\}$, we see from (3.3.61) that

$$\mathbf{A}^k \to \mathbf{A} \text{ strongly in } L^{p'(\cdot)}(\Omega_T, w^{-1/(p(\cdot)-1)}) = (L^{p(\cdot)}(\Omega_T, w))^*.$$

Hence, taking into account the convergence properties of a^k_{ij}, f_k and u_k, we conclude that $u_0 \in W^{2,1}_{p(\cdot)}(\Omega_T, w)$ is a solution of (3.0.1). The uniqueness of strong solutions of (3.0.1) directly follows from Lemma 3.3.12 and [9, Theorem 4.3].

Step2. Flattening and covering: In this subsection, we assume that the strong solution u of (3.0.1) satisfies that

$$u \in W^{2,1}_{p(\cdot)}(\Omega_T, w), \tag{3.3.64}$$

and then prove

$$\begin{aligned} &\|u_t\|_{L^{p(\cdot)}(\Omega_T,w)} + \|D^2 u\|_{L^{p(\cdot)}(\Omega_T,w)} \\ &\quad \leq c\left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)}\right). \end{aligned} \tag{3.3.65}$$

In fact, it suffices to show that

$$\|u_t\|_{L^{p(\cdot)}(\Omega_T,w)} + \|D^2u\|_{L^{p(\cdot)}(\Omega_T,w)} \leq c, \tag{3.3.66}$$

under the additional assumption that

$$\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)} \leq 1. \tag{3.3.67}$$

First, we extend the solution u and the function f in (3.0.1) to $\Omega_T^* := \Omega \times (-T, 2T)$ by letting $u(x,t) = f(x,t) = 0$ for $-T < t < 0$ and $u(x,t) = u(x, 2T-t)$, $f(x,t) = f(x, 2T-t)$ for $T < t < 2T$, and redefine the coefficient matrix $\mathbf{A}(x,t)$ by

$$\mathbf{A}(x,t) = \begin{cases} (a_{ij}(x,t)) & \text{in} \quad \mathbb{R}^n \times (-\infty, T], \\ (a_{ij}(x, 2T-t)) & \text{in} \quad \mathbb{R}^n \times (T, \infty). \end{cases}$$

Then the extended function f is obviously in $L^{p(\cdot)}(\Omega_T^*, w)$, and the redefined $\mathbf{A}$ satisfies the uniform parabolicity condition with the parabolicity constant Λ and $(4\delta, R)$-vanishing property. Furthermore, it is clear that $w \in A_{p(\cdot)}$ and we observe that u is in $W^{2,1}_{p(\cdot)}(\Omega_T^*, w)$ and solves

$$\begin{cases} u_t - a_{ij}D_{ij}u = f & \text{in} \quad \Omega_T^*, \\ u = 0 & \text{on} \quad \partial_p\Omega_T^*. \end{cases}$$

From the additional assumption (3.3.67), we also have that

$$\begin{aligned} &\|f\|_{L^{p(\cdot)}(\Omega_T^*,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T^*,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T^*,w)} \\ &\leq 2\left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)}\right) \leq 2. \end{aligned} \tag{3.3.68}$$

Now, let us fix any point $\xi = (y,s) = (y', y_n, s) \in \partial\Omega \times [0,T]$. From the boundary regularity assumption that $\partial\Omega \in C^{1,1}$, there exist $r > 0$ and a $C^{1,1}$ function $\mu = \mu(x') : \mathbb{R}^{n-1} \to \mathbb{R}$ in a new spatial coordinate system with origin at y, which is obtained by a translation and a rotation from the original one and will be still defined by x-coordinate system, such that

$$\Omega \cap B_r(0) = \{x \in B_r(0) : x_n > \mu(x')\}, \tag{3.3.69}$$

$$\mu(0) = 0, \quad \nabla_{x'}\mu(0) = 0 \quad \text{and} \quad \|\nabla^2_{x'}\mu\|_{L^\infty(\mathbb{R}^{n-1})} < \infty. \tag{3.3.70}$$

Note that (3.3.69) is also valid for all $\tilde{r} < r$ as well as r, and hence we further assume $r < \min\{T, R\}$.

In order to flatten out the boundary near the origin by changing coordinates, we define

$$\begin{cases} \tilde{x}_i &= x_i &=: \varphi^i(x), \quad \text{if } i = 1, 2, \cdots, n-1, \\ \tilde{x}_n &= x_n - \mu(x') &=: \varphi^n(x), \end{cases} \tag{3.3.71}$$

and write $\tilde{x} = \varphi(x)$. Setting $\psi := \varphi^{-1}$, we see $x = \psi(\tilde{x})$. Then we let $\tilde{\mathbf{A}}(\tilde{x}, \tilde{t}) = (\tilde{a}_{lm}(\tilde{x}, s+\tilde{t})) = [\nabla\varphi(\psi(\tilde{x}))] \cdot \mathbf{A}(\psi(\tilde{x}), s+\tilde{t}) \cdot [\nabla\varphi(\psi(\tilde{x}))]^T$, and $\tilde{p}(\tilde{x}, \tilde{t}) = p(\psi(\tilde{x}), s+\tilde{t})$. Note that $\tilde{\mathbf{A}}$ is uniformly parabolic with the parabolicity constant Λ. On the other hand, $\tilde{p}$ satisfies that $\gamma_1 \le \tilde{p}(\cdot) \le \gamma_2$ and

$$\begin{aligned} \left|\tilde{p}(\tilde{\xi}^1) - \tilde{p}(\tilde{\xi}^2)\right| &\le \theta\left(d_p((\psi(\tilde{y}^1), s+\tilde{s}^1), (\psi(\tilde{y}^2), s+\tilde{s}^2))\right) \\ &\le \theta\left((\|\nabla\psi\|_{L^\infty} + 1) d_p(\tilde{\xi}^1, \tilde{\xi}^2)\right) =: \tilde{\theta}\left(d_p(\tilde{\xi}^1, \tilde{\xi}^2)\right), \end{aligned}$$

where $\tilde{\xi}^1 := (\tilde{y}^1, \tilde{s}^1)$, $\tilde{\xi}^2 := (\tilde{y}^2, \tilde{s}^2) \in \mathbb{R}^{n+1}$ and $\tilde{\theta}(\rho) := \theta\left((\|\nabla\psi\|_{L^\infty} + 1)\rho\right)$, and hence there holds

$$\tilde{\theta}(\rho) \log\left(\frac{1}{\rho}\right) \le \tilde{M} \quad \text{for all } 0 < \rho < \infty,$$

for some constant $\tilde{M} = \tilde{M}(\mu, M) = \tilde{M}(\mu, \gamma_2, c_{LH}) > 0$.

We now choose $\rho = \rho(\rho_0, r, \mu) > 0$ so small that $Q^+_{4\rho} \subset \varphi(\Omega \cap B_r(0)) \times (-r^2, r^2)$ with $\rho \le \rho_0$ in the $(\tilde{x}, \tilde{t})$-coordinate system, where ρ_0 is given by (3.3.22), and define

$$\tilde{u}(\tilde{x}, \tilde{t}) := u\left(\psi(\tilde{x}), s+\tilde{t}\right) \quad \text{and } \tilde{w}(\tilde{x}, \tilde{t}) := w\left(\psi(\tilde{x}), s+\tilde{t}\right) \quad \text{for } (\tilde{x}, \tilde{t}) \in Q^+_{4\rho}.$$

Then we deduce that $\tilde{u}$ is in $W^{2,1}_{\tilde{p}(\cdot)}(Q^+_{4\rho}, \tilde{w})$ and solves

$$\begin{cases} \tilde{u}_{\tilde{t}} - \tilde{a}_{lm} D_{\tilde{x}_l \tilde{x}_m} \tilde{u} &= \tilde{f} \quad \text{in} \quad Q^+_{4\rho}, \\ \tilde{u} &= 0 \quad \text{on} \quad T_{4\rho}, \end{cases} \tag{3.3.72}$$

where

$$\tilde{f}(\tilde{x}, \tilde{t}) = f(\psi(\tilde{x}), s+\tilde{t}) + a_{ij}(\psi(\tilde{x}), s+\tilde{t}) \varphi^l_{x_i x_j}(\psi(\tilde{x})) D_{\tilde{x}_l} \tilde{u}.$$

From the assumption $\partial\Omega \in C^{1,1}$, we can see that $\tilde{w} \in A_{\tilde{p}(\cdot)}$ with

$$[\tilde{w}]_{A_{\tilde{p}(\cdot)}} \le c(n, \gamma_1, \gamma_2, c_{LH}, [w]_{A_{p(\cdot)}}, \mu).$$

Moreover, a direct computation yields

$$\begin{aligned}
[\tilde{\mathbf{A}}]_{4\rho} &\leq c\left([\mathbf{A}]_R + \|\nabla_{x'}\mu\|_{L^\infty(B'_r(0))} + \|\nabla_{x'}\mu\|^2_{L^\infty(B'_r(0))}\right)\\
&\leq c\left(\delta + \|\nabla_{x'}\mu\|_{L^\infty(B'_r(0))} + \|\nabla_{x'}\mu\|^2_{L^\infty(B'_r(0))}\right)\\
&\leq c\left(\delta + r\|\nabla^2_{x'}\mu\|_{L^\infty(B'_r(0))} + r^2\|\nabla^2_{x'}\mu\|^2_{L^\infty(B'_r(0))}\right)\\
&\leq c\left(\delta + r + r^2\right),
\end{aligned}$$

where we used third inequality in (3.3.70) for the last inequality.

Taking into account the conditions on f, $\mathbf{A}$ and $\partial\Omega$, it is also clear that $\tilde{f} \in L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho}, \tilde{w}\right)$ with the estimate

$$\begin{aligned}
&\|\tilde{f}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)} \\
&\leq c(\mu)\left(\|f\left(\psi(\tilde{x}), s+\tilde{t}\right)\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)} + \|D\tilde{u}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)}\right),
\end{aligned} \tag{3.3.73}$$

where $c(\mu)$ is a constant depending only on n, Λ and μ.

In turn, all the hypotheses of Theorem 3.3.13 (ii) are fulfilled with respect to the above equation (3.3.72), by taking $\delta = \delta(n, \Lambda, \gamma_1, \gamma_2, c_{LH}, \theta(\cdot), [w]_{A_{p(\cdot)}}, w(Q_{4\rho}), \mu) > 0$ and $r = r(n, \Lambda, \gamma_1, \gamma_2, \theta(\cdot), R, T, [w]_{A_{p(\cdot)}}, \mu) > 0$ sufficiently small and hence, Theorem 3.3.13 (ii) gives

$$\|\tilde{u}_{\tilde{t}}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{\rho},\tilde{w}\right)} + \|D^2\tilde{u}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{\rho},\tilde{w}\right)} \leq c\left(\|\tilde{f}\|_{L^{\tilde{p}(\cdot)}\left(Q^+_{4\rho},\tilde{w}\right)} + \|\tilde{u}\|_{L^{\gamma_1}\left(Q^+_{4\rho},\tilde{w}\right)}\right).$$

In view of (3.3.2) and (3.3.71), the change of variables from $(\tilde{x}, \tilde{t})$ to (x, t) finally yields from the previous estimate and (3.3.68) that

$$\begin{aligned}
&\|u_t\|_{L^{p(\cdot)}(V_\xi,w)} + \|D^2u\|_{L^{p(\cdot)}(V_\xi,w)} \\
&\leq c\left(\|f\|_{L^{p(\cdot)}\left(U_\xi,w\right)} + \|u\|_{L^{\gamma_1}\left(U_\xi,w\right)} + \|Du\|_{L^{p(\cdot)}\left(U_\xi,w\right)}\right)\\
&\leq c\left(\|f\|_{L^{p(\cdot)}(Q_r(\xi),w)} + \|u\|_{L^{p(\cdot)}(Q_r(\xi),w)} + \|Du\|_{L^{p(\cdot)}(Q_r(\xi),w)}\right)\\
&\leq c\left(\|f\|_{L^{p(\cdot)}(\Omega^*_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega^*_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega^*_T,w)}\right) \leq c,
\end{aligned} \tag{3.3.74}$$

where

$$V_\xi := \psi(B^+_\rho) \times \left(s - \rho^2, s + \rho^2\right)$$

and

$$U_\xi := \psi(B^+_{4\rho}) \times \left(s - (4\rho)^2, s + (4\rho)^2\right).$$

Thanks to the compactness of $\overline{\Omega_T}$, we can cover it with a finite number of sets $V_{\xi^1}, V_{\xi^2}, \cdots, V_{\xi^N}$ for some points $\xi^j \in \partial\Omega \times (0,T)$, $j = 1, 2, \cdots, N$, as above, and $V \subset\subset \Omega_T^*$ such that $\Omega_T \subset V \cup \left(\bigcup_{j=1}^N V_{\xi^j}\right)$. On the other hand, applying a standard covering argument, it follows from (3.3.18), along with (3.3.68), that

$$\begin{aligned} &\|u_t\|_{L^{p(\cdot)}(V,w)} + \|D^2u\|_{L^{p(\cdot)}(V,w)} \\ &\quad \le c\left(\|f\|_{L^{p(\cdot)}(\Omega_T^*,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T^*,w)}\right) \le c. \end{aligned} \tag{3.3.75}$$

Consequently, by summing the estimates (3.3.74) for $\xi = \xi^1, \xi^2, \ldots, \xi^N$, together with (3.3.75), we obtain (3.3.66).

Step3. Elimination of lower order terms: From (3.3.65), we have

$$\begin{aligned} &\|u\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} \\ &\le c\left(\|f\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du\|_{L^{p(\cdot)}(\Omega_T,w)}\right). \end{aligned} \tag{3.3.76}$$

It only remains to drop the last two terms on the right hand side of the previous estimate, in order to arrive at the desired estimate (3.3.7). To deal with this, we argue by contradiction. If the estimate (3.3.7) is false, then there exist sequences $\{u_k\}_{k=1}^\infty$ and $\{f_k\}_{k=1}^\infty$ such that u_k is a solution of

$$\begin{cases} (u_k)_t - a_{ij}D_{ij}u_k = f_k & \text{in} \quad \Omega_T, \\ \qquad\qquad\quad u_k = 0 & \text{on} \quad \partial\Omega_T, \end{cases}$$

satisfying

$$\|u_k\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} > k\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)}, \tag{3.3.77}$$

for any $k = 1, 2, 3, \ldots$. By an usual normalization argument, we may assume

$$\|u_k\|_{W^{2,1}_{p(\cdot)}(\Omega_T,w)} = 1. \tag{3.3.78}$$

Then (3.3.77) and (3.3.78) turn into

$$\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} < \frac{1}{k} \ \to \ 0 \quad \text{as} \quad k \to \infty. \tag{3.3.79}$$

Furthermore, by (3.3.14), (3.1.1) with $q = \tilde{\gamma}_0$ and (3.3.79), we deduce

$$\|u_k\|_{W^{2,1}_{\tilde{\gamma}_0}(\Omega_T)} \le c\|f_k\|_{L^{\tilde{\gamma}_0}(\Omega_T)} \le c\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} \ \to \ 0 \quad \text{as} \quad k \ \to \ \infty,$$

which implies that there exists a subsequence of $\{u_k\}_{l=1}^{\infty}$, still say $\{u_k\}_{k=1}^{\infty}$, such that $\lim_{k\to\infty}|u_k(z)| = \lim_{k\to\infty}|Du_k(z)| = 0$ for almost every $z \in \Omega_T$. Then Lebesgue's dominant convergence theorem along with (3.3.78) yields that

$$\int_{\Omega_T} |u_k|^{p(z)} w(z)\,dz,\ \int_{\Omega_T} |Du_k|^{p(z)} w(z)\,dz \ \to\ 0 \quad \text{as} \quad k\ \to\ \infty,$$

which means

$$\|u_k\|_{L^{p(\cdot)}(\Omega_T,w)},\ \|Du_k\|_{L^{p(\cdot)}(\Omega_T,w)} \ \to\ 0 \quad \text{as} \quad k\ \to\ \infty.$$

However, from the above result and (3.3.76), we discover

$$1 \le c\left(\|f_k\|_{L^{p(\cdot)}(\Omega_T,w)} + \|u_k\|_{L^{p(\cdot)}(\Omega_T,w)} + \|Du_k\|_{L^{p(\cdot)}(\Omega_T,w)}\right) \to 0 \text{ as } k \to \infty.$$

This contradiction establishes the desired estimates (3.3.7).

□

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국문초록

이 학위논문에서는 유계 영역에서 불연속 계수를 갖는 비발산 타원 및 포물형 방정식에 대한 최적의 정칙성 이론을 연구한다. 계수함수가 작은 BMO 조건을 가진다는 가정 하에서 그러한 방정식에 관한 디리클레 문제의 해에 대하여 가중르베그공간, 변동지수르베그공간, 가중오릭즈공간, 가중변동지수르베그공간과 같은 다양한 함수공간상에서 대역적 헤시안 가늠이 성립함을 보인다.

주요어휘: 정칙성, 비발산 타원형 방정식, 비발산 포물형 방정식, 강해, BMO 공간, 가중르베그공간, 오릭즈공간, 변동지수르베그공간
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